Chapter 1
First-order ordinary differential equations (ODEs)

1.1. Prerequisites. Formulation of engineering problems in terms of ODEs
1.2. Ordinary differential equations. Basic concepts
1.3. First-order ODEs. Initial value problem
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1.7. ODEs that reduce to exact ODEs. Integrating factors
1.8. Relaxation and equilibrium

Reading:

Kreyszig, *Advanced Engineering Mathematics, 10th Ed.,* 2011
Selection from chapter 1

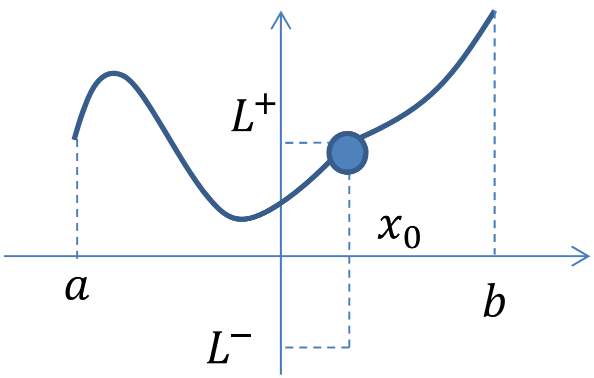
Topics for self-studying: 1.1

Continuous and discontinuous functions

Left-hand limit : $L^- = f(x_0 - 0) = \lim_{h>0, h\to 0} f(x_0 - h)$

Right-hand limit : $L^+ = f(x_0 + 0) = \lim_{h>0, h\to 0} f(x_0 + h)$

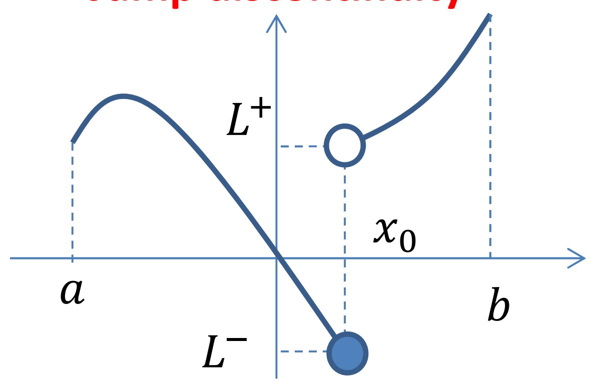
Continuous function



L^- and L^+ exist and $L^- = L^+$

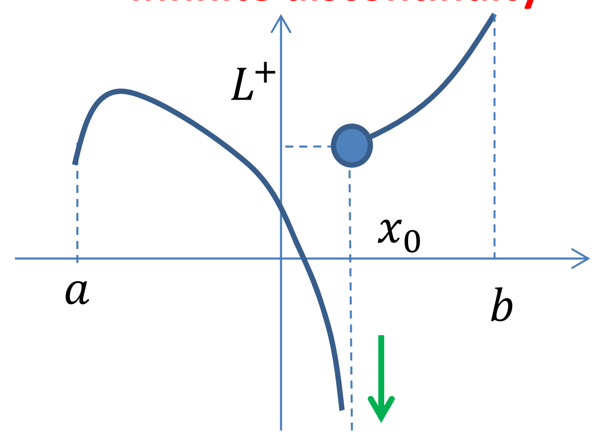
$\lim_{x\to x_0} f(x) = f(x_0) = L^- = L^+$

Discontinuous function
Jump discontinuity



L^- and L^+ exist but $L^- \neq L^+$

Discontinuous function
Infinite discontinuity



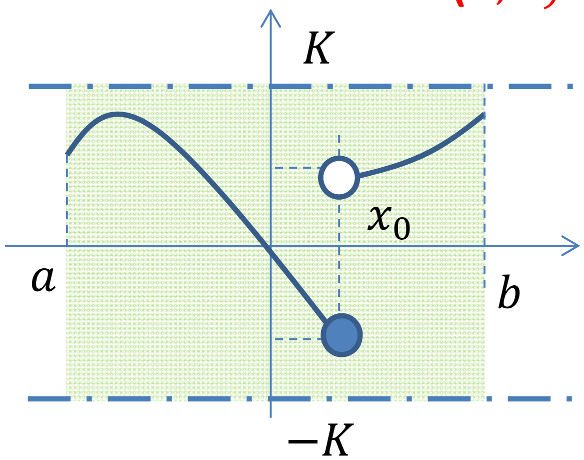
Either L^- or L^+ or both do not exist

Function $f(x)$ is **continuous in point** $x = x_0$ if the limit of $f(x)$ as x approaches x_0 through the domain of f exists and is equal to $f(x_0)$, otherwise it is called **discontinuous**.

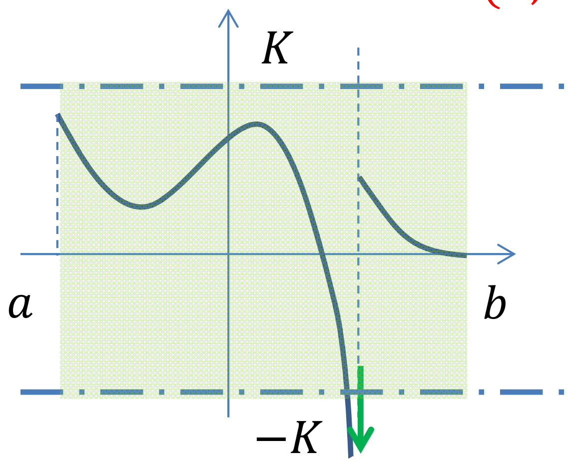
Continuous function is, roughly speaking, a function for which small changes in the input result in small changes in the output.

Bounded and differentiable functions

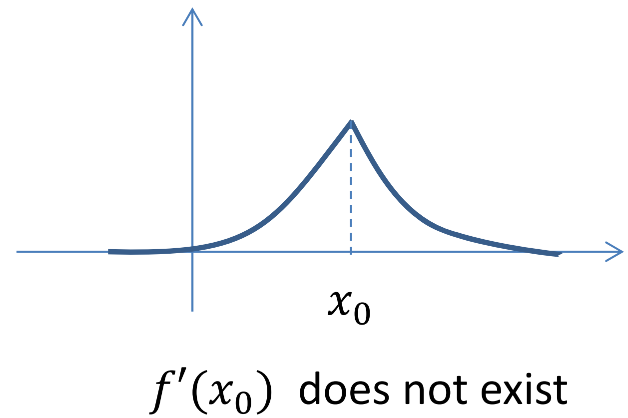
Bounded function in (a, b)



Unbounded function in (a, b)



Non-differentiable function in x_0



Function $f(x)$ is **bounded in some domain** (interval $a < x < b$) if there is a number K such that $|f(x)| \leq K$ for any point x in this domain, otherwise it is called **unbounded**.

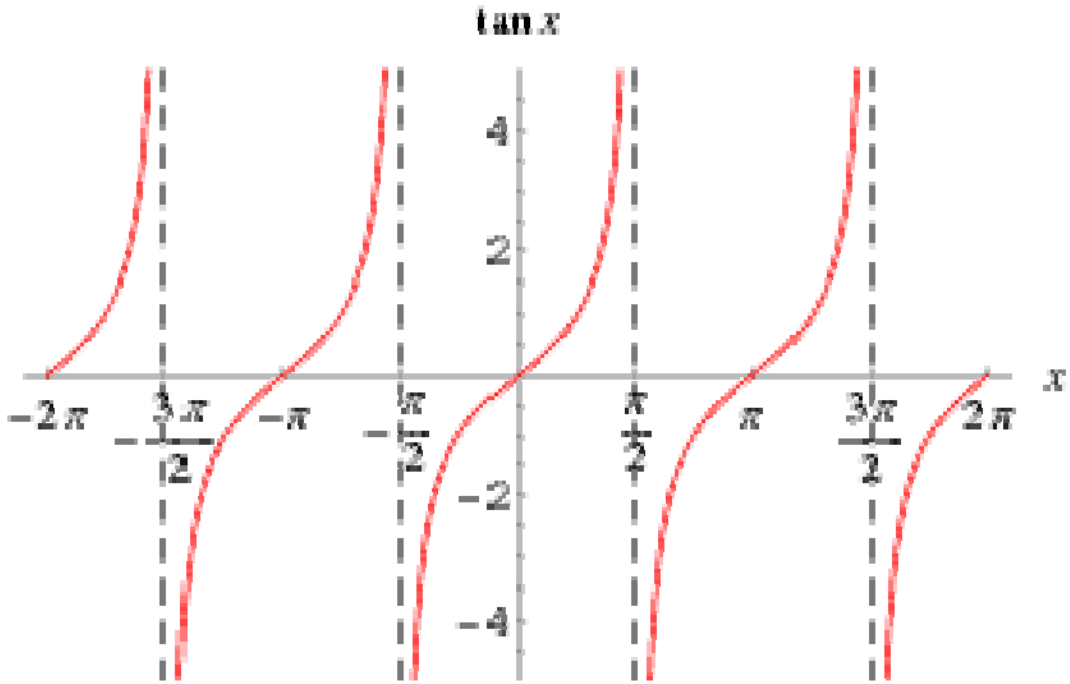
Practically it means that the absolute values of the bounded function can not be too large.

Function $f(x)$ is **differentiable in point** $x = x_0$ if the derivatives $f'(x_0)$ exists, otherwise it is called **non-differentiable**.

Differentiable function is, roughly speaking, a function for which small changes in the input result in small changes in rate of change of this function.

Function $f(x)$ is **smooth** or **infinitely differentiable** if it has derivatives of any order in any point of its domain of definition.

Example $\tan(\boldsymbol{x})$



- $y = \tan(x)$ is continuous in $x_0 = 0$.
- $y = \tan(x)$ has infinite discontinuity in $x_0 = \pm n\pi/2, \;\; n = 1, 2, 3, \ldots$
- $y = \tan(x)$ is continuous in $(-\pi/2, \pi/2)$.
- $y = \tan(x)$ is continuous in $[-\pi/4, \pi/4]$.
- $y = \tan(x)$ is bounded in $[-\pi/4, \pi/4]$.
- $y = \tan(x)$ is unbounded in $(-\pi/2, \pi/2)$.
- $y = \tan(x)$ is differentiable everywhere with exception of $x_0 = \pm n\pi/2, \;\; n = 1, 2, 3, \ldots$

Finding minimum and maximum of a function

Condition of extremum:

$$\frac{df}{dx}(x_*) = 0$$

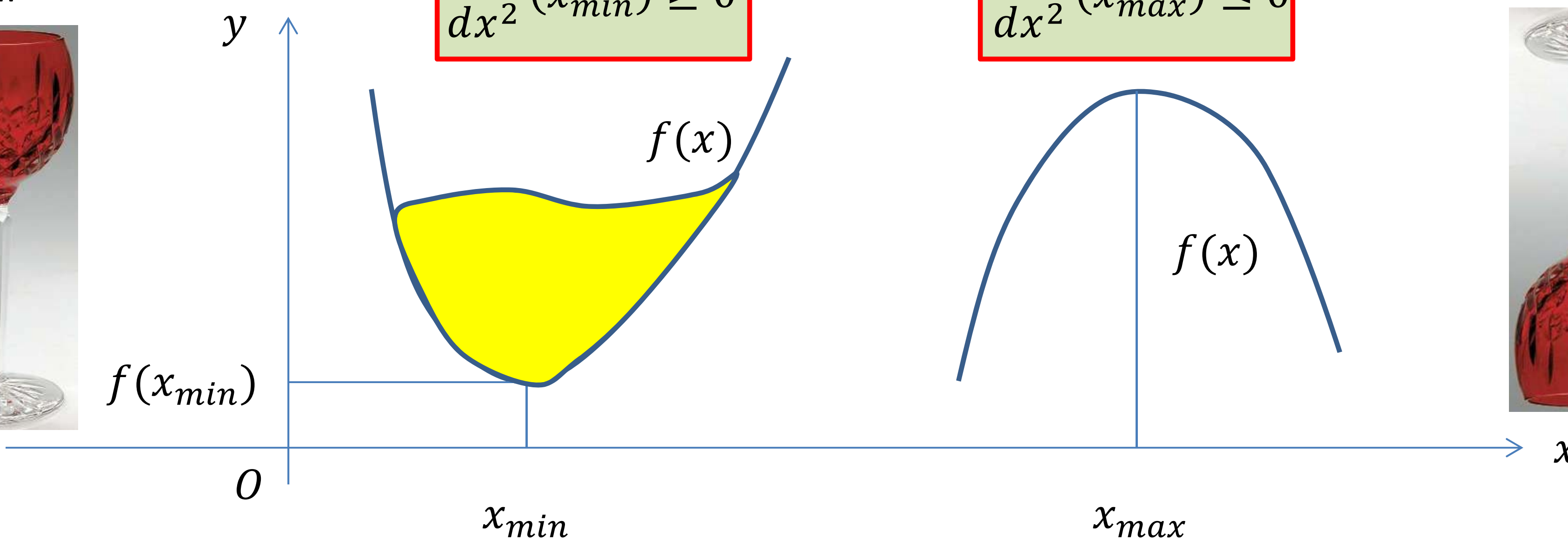
Minimum

$$\frac{d^2f}{dx^2}(x_{min}) \geq 0$$

Maximum

$$\frac{d^2f}{dx^2}(x_{max}) \leq 0$$

Full



Empty

Example: $f(x) = x^2$, $f'(x_*) = 2x_* = 0 \Rightarrow x_* = 0$ is the extremum, $f''(x_*) = 2 > 0 \Rightarrow x_* = x_{min} = 0$ is the minimum

1.1. Prerequisites. Formulation of engineering problems in terms of ODEs

Fundamental theorem of calculus

See https://en.wikipedia.org/wiki/Fundamental_theorem_of_calculus

$$F(x) = \int_a^x f(y)dy \qquad (1.1.1)$$

Here $F(x)$ is **antiderivative** of $f(x)$. Then

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(y)dy = f(x) \qquad (1.1.2)$$

$$\int_b^c f(y)dy = F(c) - F(b) \qquad (1.1.3)$$

and Eq. (1.1.1) can be re-written as

$$F(x) = \int_a^x \frac{dF}{dx}dy \qquad (1.1.4)$$

Chain rule

Let's consider two functions $f(x)$ and $g(y)$. Function $h(y) = f(g(y))$ is the **composition** of functions $f(x)$ and $g(y)$. Then

$$\frac{dh}{dy} = \frac{df}{dx}\frac{dg}{dx} \qquad (1.1.5)$$

Integration by parts

$$(fg)' = f'g + fg'$$

$$\int_a^b (fg)' dx = \int_a^b f'g dx + \int_a^b fg' dx$$

$$df = f'dx$$

$$\int_a^b fg' dx = \int_a^b (fg)' dx - \int_a^b f'g dx$$

$$f(b) - f(a) = \int_a^b f' dx$$

$$\int_a^b f dg = f(x)g(x)\Big|_{x=a}^{x=b} - \int_a^b g df \qquad (1.1.4)$$

Example:

$$\int xe^{-x} dx = -\int x de^{-x} = -xe^{-x} + \int e^{-x} dx = -(1+x)e^{-x}$$

Variable change in integrals

$$I = \int_a^b f(x)dx$$

Case I:

$$y = g(x) \qquad \Rightarrow \qquad x = g^{-1}(y), \qquad dy = g'dx$$

$$I = \int_a^b f(x)dx = \int_{g(a)}^{g(b)} \frac{f\left(g^{-1}(y)\right)}{g'(g^{-1}(y))} dy \qquad (1.1.6)$$

Case II:

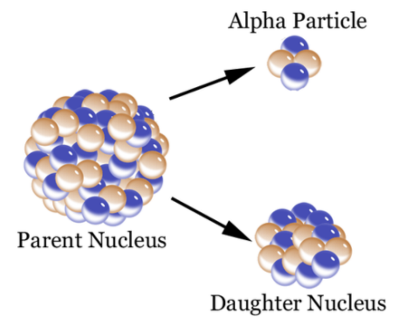
$$x = g(y) \qquad \Rightarrow \qquad y = g^{-1}(x), \qquad dx = g'dy$$

$$I = \int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f\left(g(y)\right)g'(y)dy \qquad (1.1.7)$$

Example:

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} \overset{\substack{x = \sin y \\ y = \arcsin x \\ dx = \cos y\, dy}}{=\!=\!=\!=\!=\!=} \int_{\arcsin(0)}^{\arcsin(1)} \frac{\cos y\, dy}{\sqrt{1-\sin^2 y}} = \int_0^{\pi/2} dy = \frac{\pi}{2}$$

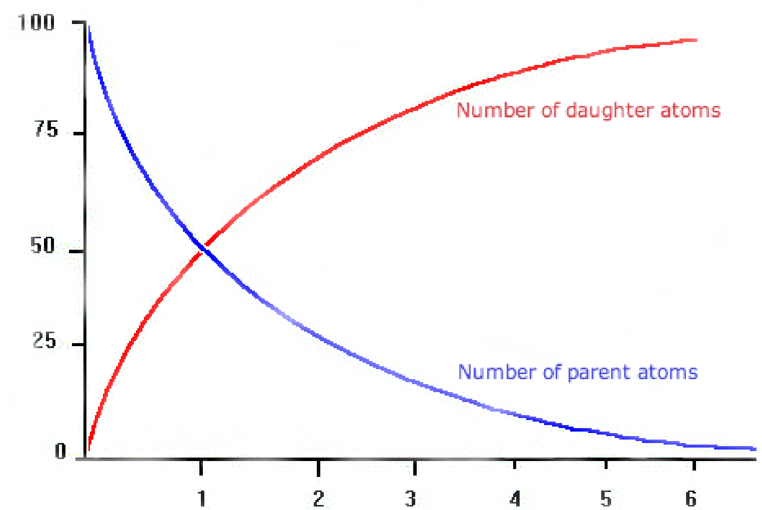
Exponential decay



Physical law: Number of nuclei exhibiting decay per unit time (radioactive decay rate) is proportional to the current number of nuclei.

$N(t)$, number of nuclei at time t

$N' = dN / dt < 0$, decay rate



1st order ODE $$dN/dt = -\lambda N$$

Solution: $$N(t) = C \exp(-\lambda t)$$

Initial condition: $N_0 = N(0)$, number of nuclei at $t = 0$

$$N(t) = N_0 \exp(-\lambda t)$$

Half-life τ is the time when a half of initial nuclei decayed

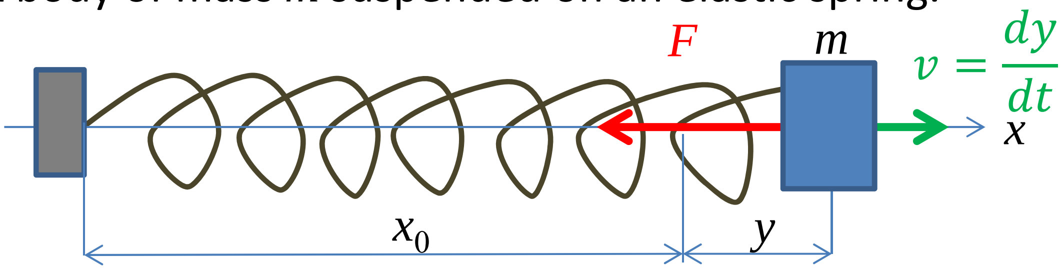
$$N(\tau) = N_0/2 = N_0 \exp(-\lambda \tau) \quad \rightarrow \quad \tau = \ln 2 / \lambda$$

Note: Exponential decay is characteristic for many physical process. For instance, friction forces between two bodies sometime result in exponential decay of their relative velocity.

1.1. Prerequisites. Formulation of engineering problems in terms of ODEs

Oscillation (Periodic motion, Cyclic motion, Vibration)

A body of mass m suspended on an elastic spring.



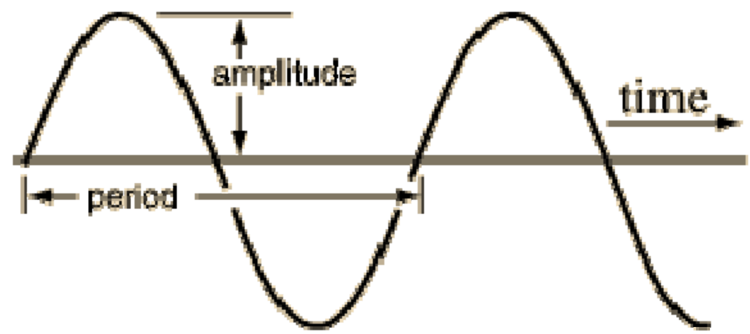
Hook's law $F = -ky$

x_0, equilibrium distance

($F = 0$ at $x = x_0$)

$y = x - x_0$, displacement

k, spring stiffness



Physical law: Newton's 2nd law of motion, $mx'' = F$

2nd order ODE: $my'' = -ky$

Solution: $y(t) = A\cos\left(2\pi\frac{t}{\tau} + \varphi\right)$

Period of oscillation: $\tau = 2\pi/\sqrt{k/m}$,

Initial conditions: $y_0 = y(0)$, $dy/dt(0) = 0$ displacement and velocity at time $t = 0$

$$y(t) = y_0\cos\left(2\pi\frac{t}{\tau}\right)$$

Note: Exponential decay and oscillation are two very general types of processes in nature.

Conclusions

- If in some problem we need to find not an individual number, but a function (e.g., coordinates of a body as functions of time), then the problem is usually mathematically formulated in terms of differential equations.
- In engineering and science, a differential equation is a mathematical formulation of a **physical law** formulated in terms of rates of change (i.e. derivatives) of some physical quantities.
- Solution of a differential equation is **not unique** (It contains arbitrary constants). It is a natural reflection of the fact that a physical law describes an infinitely large number of processes.
- In order to obtain a unique process, a unique solution of a differential equation, we need to use **additional conditions**, e.g., we need to fix initial conditions, i.e. specify the **initial state** of the process.
- The number of initial conditions (parameters which we need to fix at the initial time) coincides with the highest derivative in the equation.

1.2. Ordinary differential equations: Basic concepts

Equation = a way to formulate a mathematical problem. The solution of the problem (unknown) can be a number, a function, etc.

Differential equation is an equation

- where the unknown is a function of one or a few independent variables.
- which contains derivatives of the unknown function.

Ordinary differential equation (ODE) is a differential equation where unknown is a function of a single independent variable.

Example:

$$y' = \frac{dy}{dx} = -\lambda y$$: differential equation

x: independent variable

$y(x)$: unknown function

Partial differential equation (PDE) is a differential equation, where unknown is a function of a few independent variables.

Example:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$: Laplace equation

x, y: independent variables

$T(x, y)$: unknown function

Note: Laplace equation describes steady state temperature field $T(x, y)$ in a two-dimensional domain, where the heat conduction is governed by the Fourier law and thermal conductivity is constant.

1.2. Ordinary differential equations. Basic concepts

The general form of an ODE is

$$f\,(x, y, y', y'', y''', \ldots, y^{(n)}) = 0 \qquad (1.2.1)$$

where $y^{(n)} = d^n y \,/\, dx^n$ is the derivative of n^{th} order.

The **order** of the ODE is the highest order of derivatives in Eq. (1.2.1).

Examples:

$f\,(x, y, y') = 0$ — general form of the 1st order equation

$y' = x, \quad y' = -\lambda y, \quad (y')^2 + y = 0$

$f\,(x, y, y', y'') = 0$ — general form of the 2nd order equation

$a(x)y'' + b(x)\,y' + c(x)\,y = d(x)$ — Linear differential equation of the 2nd order

To solve an ODE means to find all functions for which the equation becomes an identity. Any such function $y = g\,(x)$ is called the **solution** of the ODE. The process of solving, i.e. finding the ODE solutions, is often called **integration**, since it usually reduces to calculation of integrals.

Note: It is easy to check whether a function $g(x)$ is the solution or not: We should substitute this function into the equation and ensure that it turns the equation into identity.

Examples:

1. $y' = h(x)$, solution $y(x) = \int h(x)dx + c$, where $c = const$.
2. $y' = x \,/\, y$, solution $x^2 - y^2 = c$.
   In order to check, let's differentiate the solution: $2x - 2yy' = 0 \;\rightarrow\; y' = x/y$.

1.2. Ordinary differential equations. Basic concepts (optional)

Explicit solution = solution in the form $y = g(x)$ (Example 1).
Implicit solution = solution in the form $G(x, y) = 0$ (Example 2).

There are a few general approaches for solving ODEs.

1. To solve an ODE **algebraically** means to represent the solution in terms of some integrals. In some cased these integrals can be further calculated in terms of elementary functions (x^n, $\exp x$, $\sin x$, etc). Analytical solutions are exact. All other types of solutions are approximate.

   Example: $y' = \exp(-x^2)$. Analytical solution $y = g(x) = \int \exp(-x^2)\, dx + c$

2. **Series solution** implies that the solution is represented as an infinite series. Although such a representation can be mathematically accurate, when calculated practically, only a finite number of members is accounted for, so it becomes approximate.

$$y = g(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \qquad \textbf{Power series}$$

3. **Iterative solutions** (**Picard method**).
4. **Numerical solutions**, when approximate solution is find in the form of a table with finite number of elements.

x_1	x_2	…	x_n
$y_1 = g(y_1)$	$y_2 = g(y_2)$	…	$y_n = g(y_n)$

5. **Asymptotic solutions** are usually obtained analyzing singular solutions of equations containing small coefficient at highest derivative.

$$ay'' + by' + cy = d(x), \qquad a \ll 1$$

1.3. First-order ODEs. Initial value problem

The general form of an ODE of the 1st order is

$$f\,(x, y, y') \;=\; 0 \qquad (1.3.1)$$

As we noted before, a solution of Eq. (1.3.1) is not unique. A set of different solutions of Eq. (1.3.1) can be written in the form

$$G(\,x, y, c\,) = 0. \qquad (1.3.2)$$

where c is an arbitrary real number. Solution in the form (1.3.2) termed **general solution**, since this equation includes a lot of solutions for different c. A **particular solution** of Eq. (1.3.1) can be obtained from its general solution if variable c is replaced by a some particular real number.

Example:

1st order ODE: $y'^2 - xy' + y = 0$

General solution: $y = cx - c^2$, family of straight lines

Check: $y' = c, c^2 - cx + cx - c^2 = 0$

Particular solution: $y = 5x - 25$

Along with particular solutions, an ODE can have **singular solutions** that cannot be obtained from Eq. (1.3.2) by varying c.

Example:

1st order ODE: $y'^2 - xy' + y = 0$

Singular solution: $y = x^2 / 4$, parabola

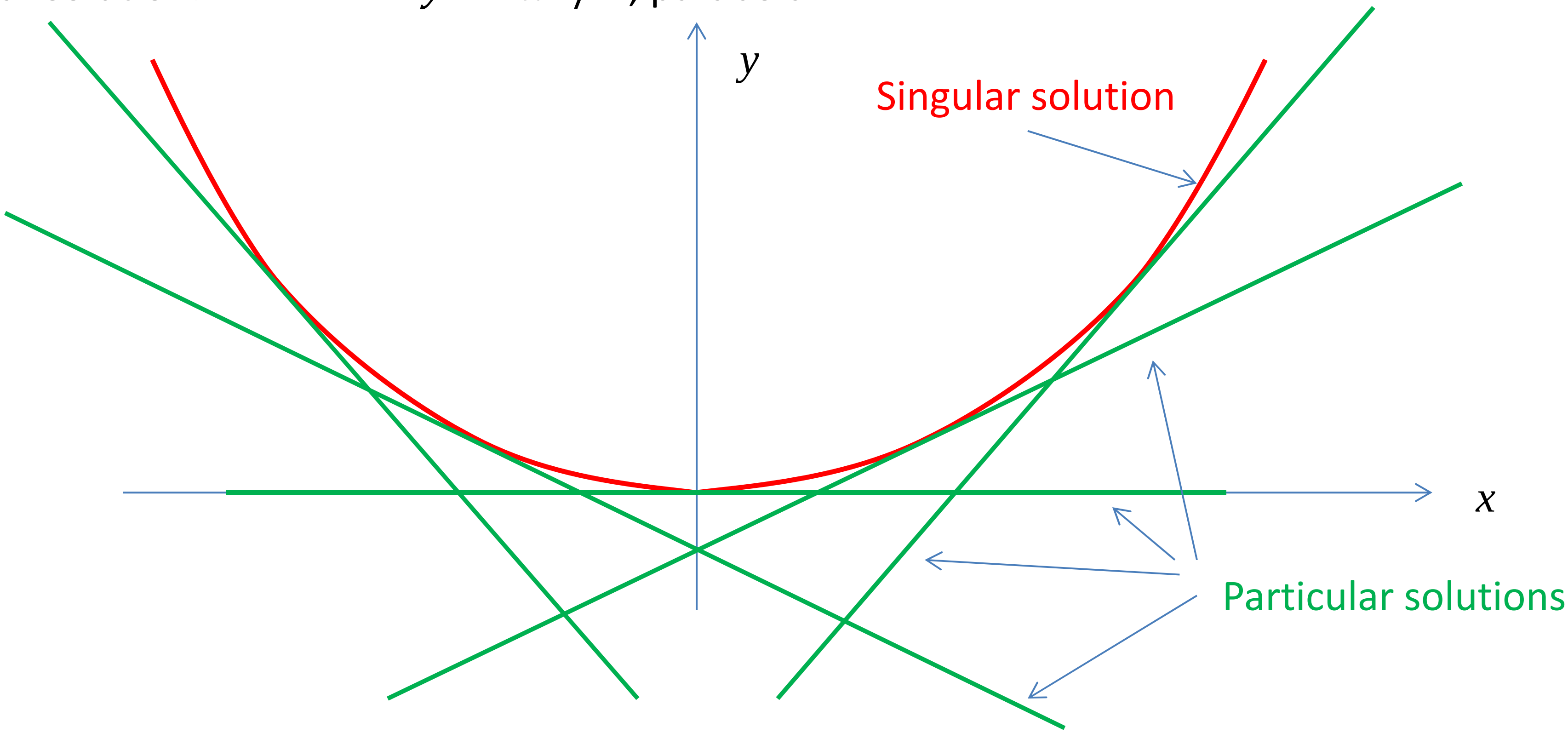
Check: $y' = x/2, \quad x^2/4 - x^2/2 + x^2/4 = 0$

1.3. First-order ODEs. Initial value problem

Geometrical representation of solutions of $y'^2 - xy' + y = 0$

General solution: $y = cx - c^2$, family of straight lines

Singular solution: $y = x^2 / 4$, parabola



Geometrical meaning of the general solutions: General solution represents a family of curves on the plane (x, y). Every particular choice of constant *c* corresponds to a particular solution and particular curve in this family. Curves representing solutions of an ODE are called **integral curves**. Some integral curves can intersect each other or reduce to a point on the plane (x, y).

1.3. First-order ODEs. Initial value problem

In many engineering (and scientific) applications we are not interested in the general solution of an ODE, but we are interested in the particular solution that satisfies some additional condition(s). For the 1st order ODE in the **explicit form** (resolved with respect to derivative)

$$y' = f(x, y) \tag{1.3.3}$$

such conditions can be formulated as a requirement that at some given point $x = x_0$ the solution $y(x)$ is equal to the prescribed value y_0, i.e.

$$y(x_0) = y_0 \qquad \text{or} \qquad y\Big|_{x=x_0} = y_0 \tag{1.3.4}$$

Eq. (1.3.4) is called the **initial condition** for Eq. (1.3.3).

A problem given by (1.3.3) and (1.3.4) is called the **initial value problem (IVP)** or **Cauchy problem**.

Formulation of the IVP: To find a particular solution of Eq. (1.3.3) that satisfies condition (1.3.4).

Note: If we are able to find the general solution of Eq. (1.3.3) algebraically in the form $G(x, y, c) = 0$, then in order to solve the initial value problem we need to find c that satisfies the equation

$$G(x_0, y_0, c) = 0$$

1.3. First-order ODEs. Initial value problem

Note: If we formulated an initial value problem, it does not necessarily mean that its solution exists and is unique. We need to know when the unique solution exists.

We should be particularly careful about existence and uniqueness of the solution of the initial value problem if we solve the problem numerically using computers.

Example 1: ODE of the 1st order: $y'^2 - xy' + y = 0$

Initial condition: $y(x_0) = y_0$

General solution: $y = cx - c^2$, family of straight lines

Let's find c:

$$y_0 = cx_0 - c^2$$

$$c^2 - x_0 c + y_0 = 0 \quad \Longrightarrow \quad c = \frac{x_0}{2} \pm \sqrt{\left(\frac{x_0}{2}\right)^2 - y_0}$$

Solution of the Cauchy problem:

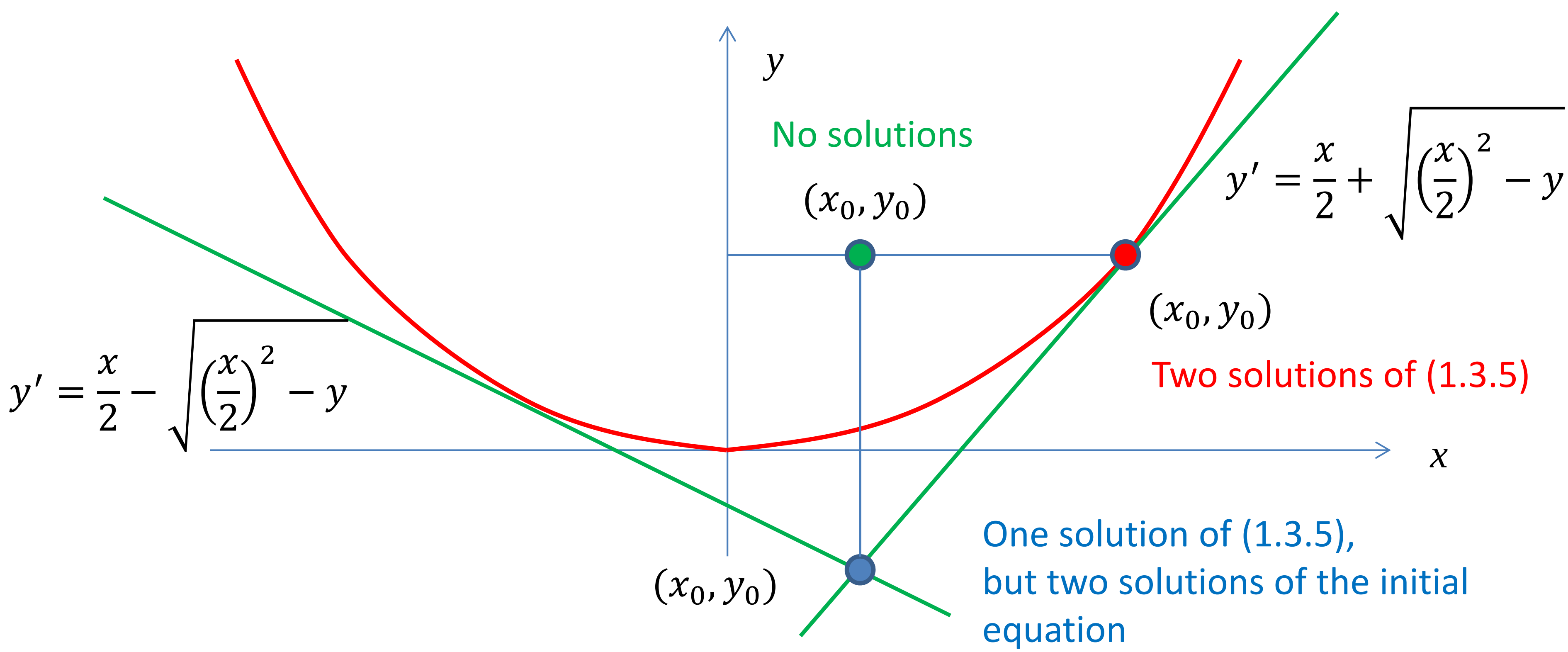
$$y(x) = \left[\frac{x_0}{2} \pm \sqrt{\left(\frac{x_0}{2}\right)^2 - y_0}\right] x - \left[\frac{x_0}{2} \pm \sqrt{\left(\frac{x_0}{2}\right)^2 - y_0}\right]^2$$

1. $(x_0/2)^2 > y_0$: Two particular solutions, Solution is not unique!
2. $(x_0/2)^2 = y_0$:One particular and one singular solution, Solution is not unique!
3. $(x_0/2)^2 < y_0$: Solution does not exist!

1.3. First-order ODEs. Initial value problem

Solution of the Cauchy problem:
$$y(x) = \left[\frac{x_0}{2} \pm \sqrt{\left(\frac{x_0}{2}\right)^2 - y_0}\right] x - \left[\frac{x_0}{2} \pm \sqrt{\left(\frac{x_0}{2}\right)^2 - y_0}\right]^2$$

ODE resolved with respect to derivative
$$y' = \frac{x}{2} \pm \sqrt{\left(\frac{x}{2}\right)^2 - y} \qquad (1.3.5)$$

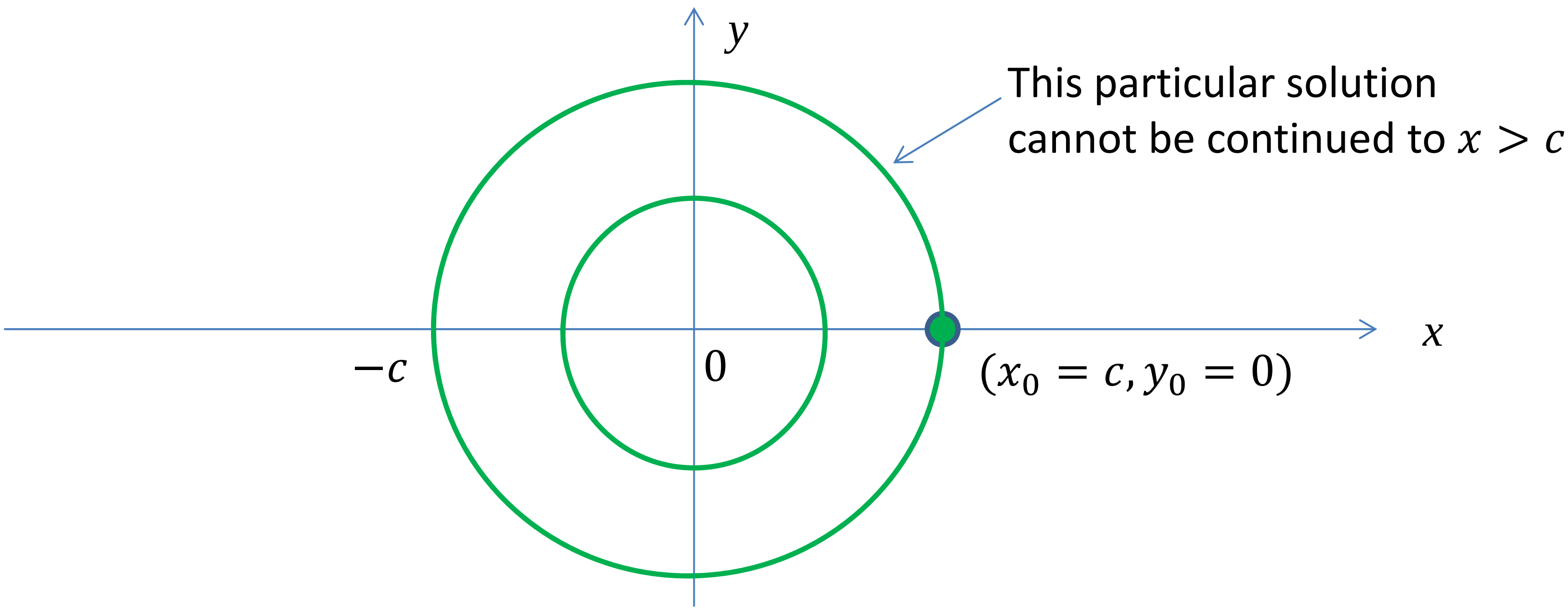


1.3. First-order ODEs. Initial value problem

Example 2: ODE of the 1st order: $y' = -x/y$

Initial condition: $y(c) = 0$

General solution: $\frac{y^2}{2} + \frac{x^2}{2} = \frac{c^2}{2}$, circles



Conclusions:

- The Cauchy problem may not have a solution at all.
- The Cauchy problem may have multiple solutions.
- Even if the Cauchy problem has a unique solution, this solution may not exist for arbitrary x.

1.3. First-order ODEs. Initial value problem (optional)

Let's consider the initial value (or Cauchy) problem for the 1st order ODE

$$y' = f(x, y) \qquad y(x_0) = y_0 \qquad (1.3.6)$$

Problem of existence: Under what conditions does an initial problem of the form (1.3.6) have at least one solution (hence one or several solutions)? Such necessary conditions are given by the **existence theorem**.

Problem of uniqueness: Under what conditions does that problem have at most one solution? Such necessary conditions are given by the **uniqueness theorem**.

In order to formulate these theorems we need to recall the notions of continuous and bounded functions.

- Function $f(x)$ is said to be **continuous in point** $x = c$ if the limit of $f(x)$ as x approaches c through the domain of f exists and is equal to $f(c)$, i.e. $\lim_{x \to c} f(x) = f(c)$.
- Function $f(x)$ is said to be **continuous in some domain** (interval $I = (a, b)$: $a < x < b$) if it is continuous in every point of this domain. Practically, it means that, for a continuous function, small changes of x corresponds to small changes of the function itself.
- Function $f(x)$ is said to be **bounded in some domain** (interval $a < x < b$) if there is a number K such that $|f(x)| \leq K$ for any point x in this domain. Practically it means, that the absolute values of the bounded function can not be too large.

Examples: $y = \tan(x)$ is continuous, but unbounded in $-\pi/2 < x < \pi/2$,
$y = \text{sign}\,(x)$ is not continuous at $x = 0$, but bounded at $-\infty < x < \infty$.

1.3. First-order ODEs. Initial value problem (optional)

Existence theorem (Peano existence theorem):

Let the RHS, $f(x, y)$, in the Cauchy problem (1.3.6)

1) be continuous at all points of some rectangle R:

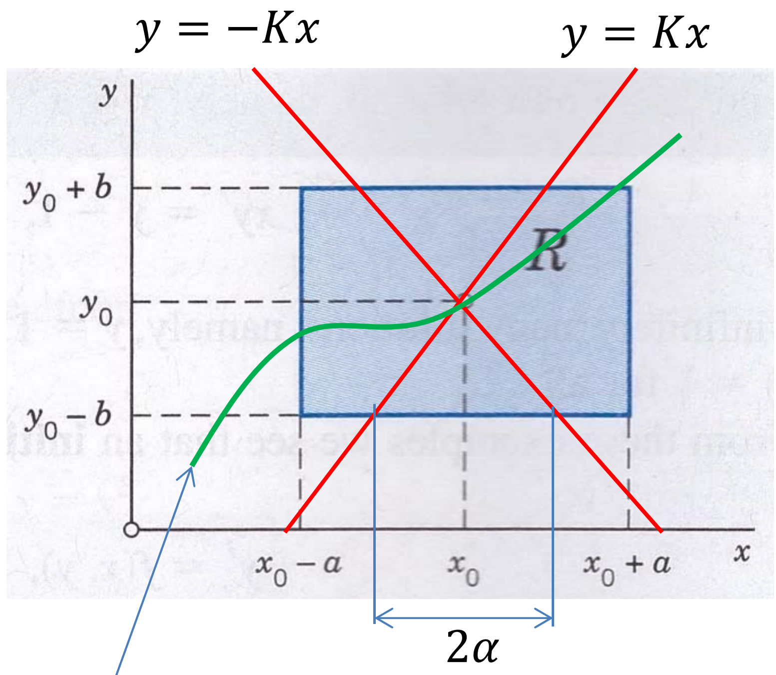
$$R: \ |x - x_0| < a, \qquad |y - y_0| < b,$$

2) and bounded in R, i.e.

$$|f(x, y)| \le K \quad \text{for all} \ (x, y) \in R$$

Then the initial value problem (1.3.6) has at least one solution $y(x)$. Such a solution exists at least for all x in the subinterval

$$|x - x_0| < \alpha = \min(a, b/K)$$



Integral curve should lay between red lines

Example:

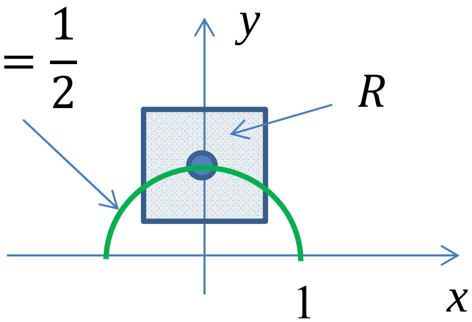
$$y' = -x/y, \qquad y(0) = 1$$

$$R: |x - x_0| < 0.5, \quad |y - y_0| < 0.5$$

$$K = \max(|f(x, y)|) = \frac{0.5}{0.5} = 1, \qquad \alpha = \min(0.5, 0.5/1) = 0.5$$

Solution exists at least at $|x - x_0| < 0.5$

$$\frac{y^2}{2} + \frac{x^2}{2} = \frac{1}{2}$$



1.3. First-order ODEs. Initial value problem (optional)

A function $f(x,y)$ is said to be **Lipschitz continuous** (or to satisfy the **Lipschitz condition** for y) in some domain G (e.g., interval $c < y < d$) if there is a constant M such that

$$\text{for all } y_1, y_2 \text{ from } G: \quad |f(x,y_2) - f(x,y_1)| \leq M|y_2 - y_1| \qquad (1.3.7)$$

Note: A function, which has a continuous and bounded derivative, $|\partial f/\partial y| \leq M$ in any point in G, satisfies the Lipschitz condition in G. It is easy to prove, using the mean value theorem:

$$|f(x,y_2) - f(x,y_1)| = \left|\int_{y_1}^{y_2} \frac{\partial f}{\partial y} dy\right| = \left|\frac{\partial f}{\partial y}\right|_{y=\tilde{y}\in G} |y_2 - y_1| \leq M\,|y_2 - y_1|$$

Examples: $f(x,y) = x^2 + y$ satisfies the Lipschitz condition;
$f(x,y) = x^2 + y^2$ does not satisfies the Lipschitz condition for $-\infty < y < \infty$.

Uniqueness theorem (Picard uniqueness theorem):

Let the RHS $f(x,y)$ in the initial problem (1.3.6) is (1) continuous, (2) bounded, and (3) satisfies the Lipschitz condition for y in some rectangle R

$$R: |x - x_0| < a, \qquad |y - y_0| < b,$$

i.e.

$$|f(x,y)| \leq K, \qquad |f(x,y_2) - f(x,y_1)| \leq M|y_2 - y_1| \qquad \text{for all } (x,y) \in R$$

Then the initial value problem (1.3.6) has a unique solution. This solutions exists at least at

$$|x - x_0| < \alpha = \min(a, b/K)$$

1.3. First-order ODEs. Initial value problem (optional)

Note: *The Lipschitz condition (or bounded derivative $\partial f/\partial y$) is essential for uniqueness.* Only the existence of derivative $\partial f/\partial y$ is not enough for the uniqueness of the solution of the initial value problem.

Example: Initial value problem $y' = \sqrt{|y|}$, $y(0) = 0$ has two solutions

$$y = 0 \qquad \text{and} \qquad y^* = \begin{cases} x^2/4 & \text{if } x \geqq 0 \\ -x^2/4 & \text{if } x < 0 \end{cases}$$

although $f(x, y) = \sqrt{|y|}$ is continuous for all y. The Lipschitz condition (4) is violated in any region that includes the line $y = 0$, because for $y_1 = 0$ and positive y_2 we have

$$\frac{|f(x, y_2) - f(x, y_1)|}{|y_2 - y_1|} = \frac{\sqrt{y_2}}{y_2} = \frac{1}{\sqrt{y_2}}, \qquad (\sqrt{y_2} > 0)$$

1.4. Separable ODEs

Separable ODE or ODE **with separable variables** is an ODE which can be written in the form

$$g(y)y' = f(x)$$

or, since $y' = dy/dx$

$$g(y)dy = f(x)dx \qquad (1.4.1)$$

It is important that g depends only on y and f depends only on x in Eq. (1.4.1).

The general solution of Eq. (1.4.1) can be found by integrating left- and right-hand sides of this equation and can be written in the form

General solution of the separable ODE

$$\int g(y)dy = \int f(x)dx + c \qquad (1.4.2)$$

Note 1: In some cases, even if a equation does not look like Eq. (1.4.1), it can be easily reduced to the form of Eq. (1.4.1).

1. Equation $f(x)\,dy = g(y)\,dx$ reduces to $dy/g(y) = dx/f(x)$.

2. Equation $f_1(x)g_1(y)\,dy + f_2(x)g_2(y)\,dx = 0$ is also separable if $f_1(x)g_2(y) \neq 0$, since then it reduces to

$$\frac{g_1(y)}{g_2(y)}dy = -\frac{f_2(x)}{f_1(x)}dx$$

1.4. Separable ODEs

Note 2: Geometrically, the general solution corresponds to a family of integral curves on the plane (x, y). Any particular choice of parameter c in Eq. (1.4.1) corresponds to an integral curve.

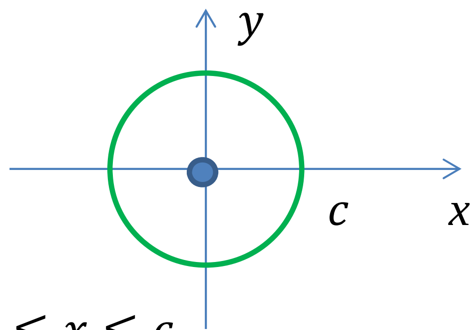
Examples: Equation of type $x^k y^l dy \pm x^m y^n dx = 0, \quad k, l, m, n = 0, \pm 1$

1

$$y dy + x dx = 0$$

$$\frac{y^2}{2} + \frac{x^2}{2} = \frac{c^2}{2}$$

Circle $\quad -c \le x \le c$

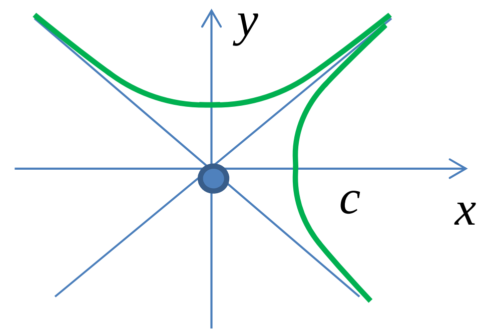


2

$$y dy - x dx = 0$$

$$\frac{y^2}{2} - \frac{x^2}{2} = \pm \frac{c^2}{2}$$

Hyperbole



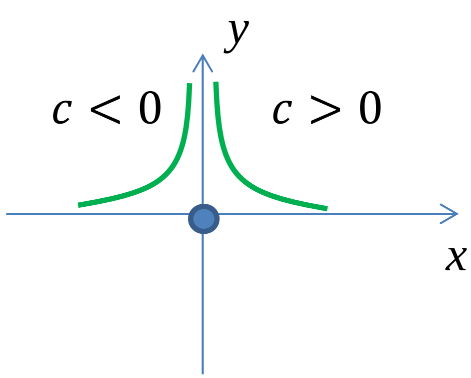
3

$$x dy + y dx = 0$$

$$\ln y + \ln x = \ln c$$

$$xy = c$$

Hyperbole



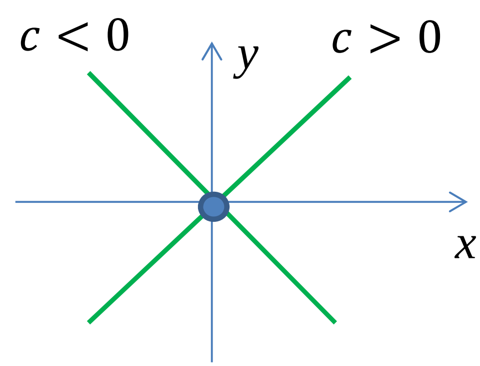
4

$$x dy - y dx = 0$$

$$\ln y - \ln x = \ln c$$

$$y = cx$$

Line



Note: In examples 1-4, $x = 0, y = 0$ is the singular point = singular solution.

1.4. Separable ODEs

For the 1st order ODE in the **symmetrical form**

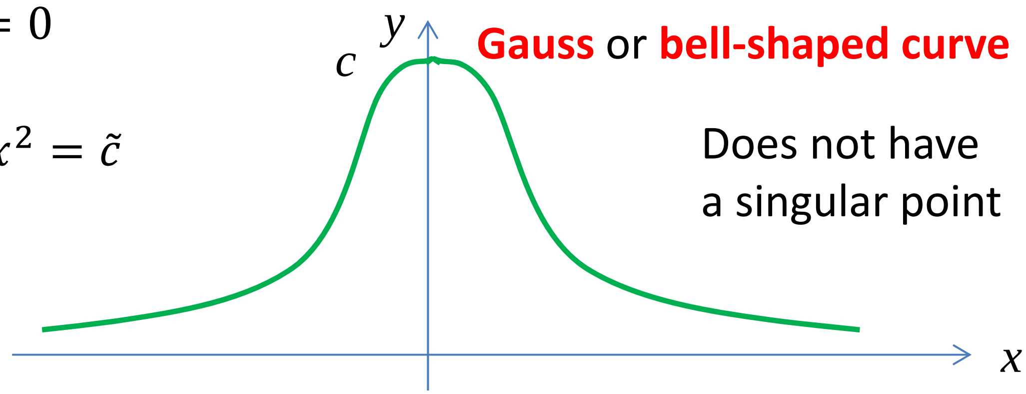
$$M(x,y)dx + N(x,y)dy = 0$$

the **singular point** (x_0, y_0) is a point where $M\ (x_0, y_0)\ =\ N\ (x_0, y_0)\ =\ 0.$

Example 5: $y' = -2xy \implies \dfrac{dy}{y} + 2xdx = 0$

$$\int \frac{dy}{y} + \int 2xdx = \tilde{c} \implies \ln y + x^2 = \tilde{c}$$

$$y = \exp\ (\tilde{c} - x^2)$$
$$y = c \exp(-x^2)$$



Gauss or **bell-shaped curve**

Does not have a singular point

Note 1: Gauss curve is important function in many applications

- Probability theory (Gaussian distribution of a random variable).
- Statistical physics and kinetic theory of gases (Maxwell-Boltzmann velocity distribution).

Note 2: These examples show that different particular solutions are defined in different domains (intervals) of independent variable x (example 1).

Note 3: We should be careful in derivation of general solutions: Operations like division, taking square root or ln can result in the “loss” of some particular solutions (example 4).

Example 6: Gas compression in a piston

Many problems in thermodynamics reduce to separable ODEs

due to the specific form of the 1st law of thermodynamics

$$\delta Q = dU + pdV$$

Let's consider a fixed amount of an ideal gas (N molecules) in the piston, then

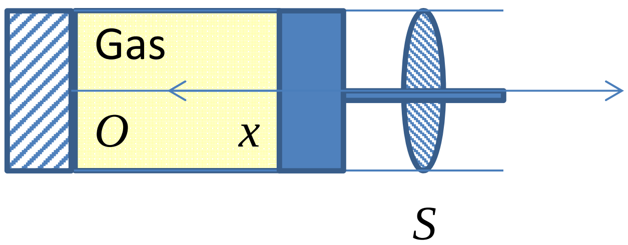
$$dU = C_v dT, \qquad p = \frac{NkT}{V}, \qquad V = Sx, \qquad dV = Sdx$$

Assume that compression/decompression occurs adiabatically, i.e. $\delta Q = 0$:

$$C_v dT + NkT\frac{dx}{x} = 0$$

$$\frac{dT}{T} = -\frac{Nk}{C_v}\frac{dx}{x} = -a\frac{dx}{x} \quad : \text{Separable Equation}$$

$$Tx^a = c$$



$$\text{Initial value problem: } T(x_0) = T_0$$

$$T = T_0\left(\frac{x_0}{x}\right)^a$$

Compression → Temperature rises
Decompression → Temperature drops

1.4. Separable ODEs

Some equations can be reduced to the separable form by the change of variable.

A function $h\ (x, y)$ is called a **homogeneous function of degree *k*** if

$$h(\lambda x, \lambda y) = \lambda^k h(x, y) \qquad (1.4.3)$$

Examples: $h = xy, h = x^2 + y^2, h = xy + x^2$ are homogeneous functions of degree 2

Let's consider an ODE in the symmetrical form

$$G(x, y)dy + F(x, y)dx = 0 \qquad (1.4.4)$$

where $G(x, y)$ and $F(x, y)$ are homogeneous functions of the same degree. We can prove that

1. An ODE (1.4.4), where $G(x, y)$ and $F(x, y)$ are homogeneous functions of the same degree, reduces to equation

$$y' = g\left(\frac{y}{x}\right) \qquad (1.4.5)$$

Proof:

$$(1.4.4) \Longrightarrow y' = \frac{dy}{dx} = -\frac{F(x, y)}{G(x, y)} = -\frac{F\left(x \cdot 1, x \cdot \left(\frac{y}{x}\right)\right)}{G\left(x \cdot 1, x \cdot \left(\frac{y}{x}\right)\right)} \underset{\lambda = x}{\overset{(1.4.3)}{=}} -\frac{x^k F\left(1, \frac{y}{x}\right)}{x^k G\left(1, \frac{y}{x}\right)} = -\frac{F\left(1, \frac{y}{x}\right)}{G\left(1, \frac{y}{x}\right)} \Longrightarrow (1.4.5)$$

2. Any equation in the form (1.4.5) reduces to a separable ODE.

Proof: Lets introduce new variable $u(x) = y\ (x)/x$ or $y(x) = u(x)x$. Then

$$(1.4.5) \Longrightarrow \frac{dy}{dx} = \frac{du}{dx}x + u = g\left(\frac{y}{x}\right) \Longrightarrow xu' = g(u) - u \Longrightarrow \qquad \frac{du}{g(u) - u} = \frac{dx}{x}$$

2.4. Separable ODEs

Example:

$$y' = \frac{x+y}{x-y}$$

$$y' = \frac{1+y/x}{1-y/x} = g\left(\frac{y}{x}\right)$$

$$u = \frac{y}{x} \quad \Longrightarrow \quad \frac{du}{\frac{1+u}{1-u} - u} = \frac{dx}{x}$$

$$(1-u)\frac{du}{1+u^2} = \frac{dx}{x}$$

$$\int \frac{du}{1+u^2} - \int \frac{udu}{1+u^2} = \int \frac{dx}{x}$$

$$\arctan u - \frac{1}{2}\ln(u^2+1) = \ln x + c$$

$$\arctan \frac{y}{x} - \frac{1}{2}\ln\left[\left(\frac{y}{x}\right)^2 + 1\right] = \ln x + c$$

1.5. Linear ODEs

An ODE of the 1st order is said to be **linear** if it can be written as

$$y' + p(x)y = r(x) \tag{1.5.1}$$

where $p(x)$ and $r(x)$ are arbitrary functions of independent variable x only. If $r(x) = 0$, then Eq. (1.5.1) is said to be **homogeneous** equation, otherwise it is said to be **nonhomogeneous**.

Note 1: Existence and uniqueness theorem. For any ODE (1.5.1) with continuous $p(x)$ and $r(x)$ in $|x - x_0| \leq a$, the Lipschitz condition holds and *the Cauchy problem has a unique solution.*

Proof:

$$f(x, y) = r(x) - p(x)y \quad \Longrightarrow \quad |f(x, y_2) - f(x, y_1)| = |p(x)||y_2 - y_1| \leq M|y_2 - y_1|$$

since continuous function $p(x)$ in the closed and bounded domain, $|x - x_0| \leq a$, is bounded.

Note 2: Linear homogeneous equation is also separable and, thus, can be solved algebraically by the method developed for separable ODEs.

Note 3: Solutions of linear homogeneous ODEs possess the following property: if $y = g_1(x)$ and $y = g_2(x)$ are two solutions, then $y(x) = c_1\, g_1(x) + c_2\, g_2(x)$ is also a solution (c_1 and c_2 are arbitrary constants).

Note 4: Any linear ODE can be solved algebraically. Two equivalent ways:

1. To find an integrating factor and reduce Eq. (1.5.1) to an exact ODE: See Kreyszig, Sect. 1.5.
2. To reduce Eq. (1.5.1) to a separable ODE.

2.5. Linear ODEs

Let's look for a solution in the form

$$y(x) = u(x)v(x) \qquad (1.5.2)$$

and substitute Eq. (1.5.2) into Eq. (1.5.1). Then

$$u'v + uv' + puv = r \qquad (1.5.3)$$

We can choose such $v(x)$ that the sum of 2nd and 3rd terms in the LHS of Eq. (2.5.3) turns to 0:

$$v' + pv = 0 \;\Longrightarrow\; v' = -p(x)v \;\Longrightarrow\; \frac{dv}{v} = -p(x)dx \;\longrightarrow\; \text{separable ODE}$$

$$v = \exp\left(-\int p(x)dx\right) \qquad (1.5.4)$$

Now let's insert Eq. (1.5.4) into left-hand part of Eq. (1.5.3):

$$u' \exp\left(-\int p(x)dx\right) = r(x) \;\Longrightarrow\; u' = r(x)\exp\left(\int p(x)dx\right) \;\longrightarrow\; \text{separable ODE}$$

$$u = \int r(x)\exp\left(\int p(x)dx\right)dx + c \quad + \quad \text{Eq. (1.5.4)} \;\Longrightarrow\; \text{Eq. (1.5.2)}$$

General solution of the 1st order linear ODE

$$y = uv = \exp\left(-\int p(x)dx\right)\left[\int r(x)\exp\left(\int p(x)dx\right)dx + c\right] \qquad (1.5.5)$$

1.5. Linear ODEs

The general solution can be written in a slightly different form

$$h(x) = \int p(x)dx \qquad y = \exp(-h(x))\left[\int r(x)\exp(h(x))\,dx + c\right] \qquad (1.5.6)$$

Example 1: $y' = -2xy$ is linear homogeneous ODE ($p = 2x, r = 0$),
solution is the Gauss curve

Example 2: $y' = -2xy + xe^{-x^2}$: Linear ODE: $p(x) = 2x$, $r(x) = xe^{-x^2}$

$$v(x) = \exp\left(-\int p(x)dx\right) = \exp\left(-\int 2xdx\right) = e^{-x^2}$$

$$u(x) = \int r(x)\exp\left(\int p(x)dx\right)dx + c = \int xe^{-x^2}e^{x^2}dx + c = \frac{x^2}{2} + c$$

$$y(x) = u(x)v(x) = e^{-x^2}\left(\frac{x^2}{2} + c\right)$$

1.5. Linear ODEs

Example 3: Some non-linear equations reduce to linear ones by the change of variable.

$$(x^2+1)(2xdx + \cos y\, dy) = -2x \sin y\ dx \quad \text{: Non-linear equation}$$

Change of variable: $z = \sin y\,, \quad dz = \cos y\ dy$

$$2x + z' = -\frac{2x}{x^2+1} z \quad \text{: Linear ODE:} \quad p(x) = \frac{2x}{x^2+1}, \qquad r(x) = -2x$$

$$h(x) = \int p(x)dx = \int \frac{2xdx}{x^2+1} = \int \frac{d(x^2+1)}{x^2+1} = \ln\,(x^2+1)$$

$$v(x) = e^{-h(x)} = \frac{1}{x^2+1}$$

$$u(x) = \int r(x)e^{h(x)}dx + c = \int (-2x)(x^2+1)dx + c = -\frac{x^4}{2} - x^2 + c$$

$$z(x) = u(x)v(x) = \frac{c - x^2(x^2/2+1)}{x^2+1}$$

$$y(x) = \arcsin z = \arcsin\left[\frac{c - x^2(x^2/2+1)}{x^2+1}\right]$$

1.5. Linear ODEs

The 1st order ODE

$$y' + p(x)y = g(x)y^a \qquad (1.5.7)$$

where a is an arbitrary real number, is called the **Bernoulli equation**.

At $a = 0$ and $a = 1$ the Bernoulli equation is linear, otherwise is nonlinear, but it reduces to a linear equation by the change of variable.

Let's first transform Eq. (1.5.7) into

$$\frac{y'}{y^a} + \frac{p(x)}{y^{a-1}}(x) = g(x)$$

and then introduce new variable u:

$$u = y^{1-a} \quad \Longrightarrow \quad u' = (1-a)y^{-a}y' \quad \Longrightarrow \quad \frac{y'}{y^a} = \frac{u'}{1-a}$$

$$\frac{u'}{1-a} + p(x)u = g(x)$$

$$u' + (1-a)p(x)u = (1-a)g(x) \qquad \text{: Linear ODE}$$

Problem: A specific case of the Bernoulli equation is the **logistic equation** $y' = Ay - By^2$ (A and B are arbitrary constants), which corresponds to a simple model of the population dynamics. Find a general solution of this equation.

1.6. Exact ODEs

The 1st order ODE in the symmetrical form

$$M(x,y)dx + N(x,y)dy = 0 \qquad (1.6.1)$$

is said to be **exact** if there is such function $u = u(x,y)$ that the left-hand side if Eq. (1.6.1) is the total (or exact) differential of u, i.e. can be represented in the form

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad (1.6.2)$$

If Eq. (1.6.1) is exact then $du = 0$, which means that the general solution of exact Eq. (1.6.1) takes the form

$$\boxed{u(x,y) = c} \qquad (1.6.3)$$

Comparing (1.6.1) and (1.6.2) one can conclude that for the exact equation

$$M(x,y) = \frac{\partial u}{\partial x} \qquad N(x,y) = \frac{\partial u}{\partial y}$$

Assume that M and N have continuous first derivatives. Then

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

If mixed derivatives are continuous, then they are equal to each other, i.e.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad (1.6.4)$$

2.6. Exact ODEs

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \textbf{Criterion of exactness} \qquad (1.6.4)$$

Thus, condition (1.6.4) is *necessary* for the left-hand size of Eq. (1.6.1) to be the total differential (necessity means that (1.6.2) results in (1.6.4) for arbitrary M and N). Let's prove that the condition (criterion) given by Eq. (1.6.4) is *sufficient*, i.e. if arbitrary M and N satisfy (1.6.4), then $u(x, y)$ exists which satisfy Eq. (1.6.2) and, thus, Eq. (1.6.1) is exact.

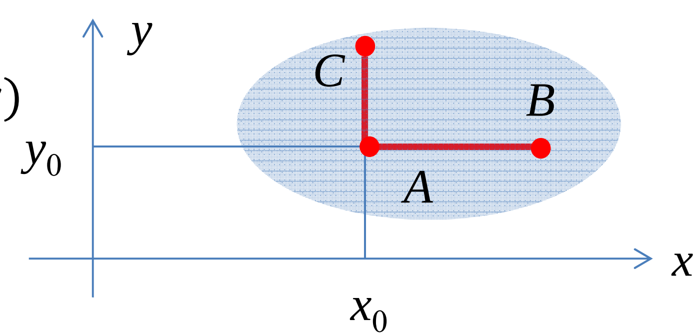
Plan of the proof:

1. To construct a function $u(x, y)$ that satisfies $\partial u/\partial x = M(x, y)$ and $\partial u/\partial y = N(x, y)$.
2. To show that $u(x, y)$ becomes a solution if the criterion given by Eq. (1.6.4) is satisfied.

Let's chose some point (x_0, y_0) on plane (x, y) such that it belongs to a particular solution of Eq. (1.6.1) and in vicinity of this point any point also belongs to some particular solution.

Then let's introduce a function $u(x, y)$ which satisfy equation $\partial u/\partial x = M(x, y)$. Then the solution of the last equation along the path AB can be written in the form

$$u(x, y) = \int_{x_0}^{x} \frac{\partial u}{\partial x} dx + k(y) = \int_{x_0}^{x} M(x, y) dx + k(y) \qquad (1.6.5)$$



If we do it at different y (taking different paths AB), then the "constant" k depends on y.

1.6. Exact ODEs

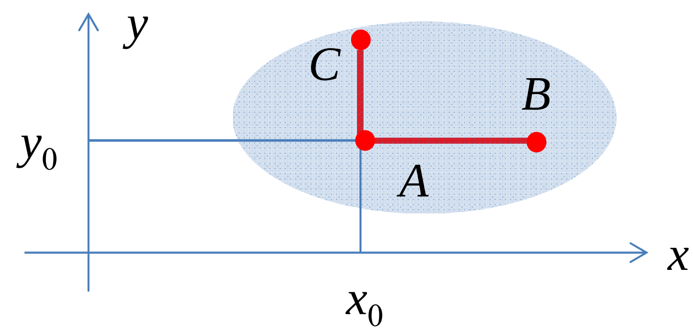
$$u(x,y) = \int_{x_0}^{x} M(x,y)dx + k(y) \qquad (1.6.5)$$

Now let's find $k(y)$ that satisfies equation $\partial u/\partial y = N(x,y)$:

$$\frac{\partial u}{\partial y} = \int_{x_0}^{x} \frac{\partial M}{\partial y} dx + \frac{dk}{dy} = N(x,y)$$

$$\int_{x_0}^{x} \frac{\partial N}{\partial x} dx + \frac{dk}{dy} = N(x,y) \implies N(x,y) - N(x_0,y) + \frac{dk}{dy} = N(x,y)$$

$$\frac{dk}{dy} = N(x_0,y)$$



Now let's integrate the obtained equation along path AC

$$k(y) = \int_{y_0}^{y} N(x_0,y)dy + a$$

choose the particular solution corresponding to $a = 0$, and substitute $k(y)$ into Eq. (1.6.5)

$$u(x,y) = \int_{x_0}^{x} M(x,y)dx + \int_{y_0}^{y} N(x_0,y)dy \qquad (1.6.6)$$

1.6. Exact ODEs

$$u(x,y) = \int_{x_0}^{x} M(x,y)dx + \int_{y_0}^{y} N(x_0,y)dy$$

Now we can prove that total differential of $u(x,y)$ given by Eq. (1.6.6) coincides with LHS of Eq. (1.6.1) and, thus, condition (1.6.4) implies that Eq. (1.6.1) is exact.

Here we use the fundamental theorem of calculus, Eq. (1.1.2)

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = M(x,y)dx + \left[\int_{x_0}^{x} \frac{\partial M}{\partial y}dx + N(x_0,y)\right]dy \overset{(1.6.4)}{=}$$

$$M(x,y)dx + \left[\int_{x_0}^{x} \frac{\partial N}{\partial x}dx + N(x_0,y)\right]dy = M(x,y)dx + [N(x,y) - N(x_0,y) + N(x_0,y)]dy =$$

Here we use Eq. (1.1.4)

$$= M(x,y)dx + N(x,y)dy$$

Eqs. (1.6.6) together with (1.6.3) give the general solution of exact Eq. (1.6.1).

General solution of the exact ODE

$$\int_{x_0}^{x} M(x,y)dx + \int_{y_0}^{y} N(x_0,y)dy = c \tag{1.6.7}$$

1.6. Exact ODEs

Example 1:

$$y dx + x dy = 0$$

$$\frac{\partial M}{\partial y} = 1, \qquad \frac{\partial N}{\partial x} = 1 \implies \text{Equation is exact}$$

$$u(x,y) = \int_{x_0}^{x} y dx + \int_{y_0}^{y} x_0 dy = y(x - x_0) + (y - y_0)x_0 = xy - y_0 x_0 \implies$$

$$\text{General solution:} \quad xy = c$$

Example 2:

$$e^{-y} dx - (2y + xe^{-y}) dy = 0$$

$$\frac{\partial M}{\partial y} = -e^{-y}, \qquad \frac{\partial N}{\partial x} = -e^{-y} \implies \text{Equation is exact}$$

$$u(x,y) = \int_{x_0}^{x} e^{-y} dx - \int_{y_0}^{y} (2y + x_0 e^{-y}) dy =$$

$$= e^{-y}(x - x_0) - (y^2 - y_0^2) + x_0(e^{-y} - e^{-y_0}) = \; xe^{-y} - y^2 + y_0^2 - x_0 e^{-y_0} \implies$$

$$\text{General solution:} \quad xe^{-y} - y^2 = c$$

1.7. ODEs that reduce to exact ODEs. Integrating factors

Assume that a 1st order ODE in the symmetrical form

$$P(x,y)dx + Q(x,y)dy = 0 \qquad (1.7.1)$$

is not exact. Sometime it is possible to transform then such equation into the exact form after multiplying by a suitable function $F(x,y) \not\equiv 0$. If Eq. (1.7.2)

$$F(x,y)P(x,y)dx + F(x,y)Q(x,y)dy = 0 \qquad (1.7.2)$$

is exact, then function $F(x,y)$ is termed the **integrating factor** for Eq. (1.7.1).

In order to find an integrating factor, one should use the criterion of exactness,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \Longrightarrow \quad \frac{\partial (FP)}{\partial y} = \frac{\partial (FQ)}{\partial x} \qquad (1.7.3)$$

i.e. the integrating factor should be a solution of the *partial* differential equation

$$\frac{\partial F}{\partial y}P + F\frac{\partial P}{\partial y} = \frac{\partial F}{\partial x}Q + F\frac{\partial Q}{\partial x}$$

or

$$\frac{1}{F} = \frac{\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}}{Q\dfrac{\partial F}{\partial x} - P\dfrac{\partial F}{\partial y}} \qquad (1.7.4)$$

If integrating factor $F(x,y)$ exists, it must satisfy this equation

We cannot find a solution of (1.7.4). We even do not know whether the solution exists or not.

1.7. ODEs that reduce to exact ODEs. Integrating factors

There is no rule that can give us an integrating factor for any ODE. We can find the integrating factor, if we reduce Eq. (1.7.4) to an ODE. The following theorem shows us how it can be done.

Theorem:

Let's consider a 1st order ODE in the symmetrical form, Eq. (1.7.1), and assume that we found such function $\omega(x,y)$ that

1. $\omega(x,y)$ has continuous derivatives $\partial\omega/\partial x$ and $\partial\omega/\partial y$.
2. $\omega(x,y)$ satisfies the equation

$$\chi(\omega) = \frac{\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}}{Q\dfrac{\partial \omega}{\partial x} - P\dfrac{\partial \omega}{\partial y}} \qquad (1.7.5)$$

Then the integrating factor exists and is equal to

F is the composition of functions μ and ω

$$F(x,y) = \mu(\omega(x,y)), \qquad \mu(\omega) = \exp[\int \chi(\omega) d\omega] \qquad (1.7.6)$$

Proof:

Let's look for $F(x,y)$ in the form $F(x,y) = \mu(\omega(x,y))$. Then

Here we use the chain rule, Eq. (1.1.5)

$$\frac{\partial F}{\partial x} = \frac{d\mu}{d\omega}\frac{\partial \omega}{\partial x}, \qquad \frac{\partial F}{\partial y} = \frac{d\mu}{d\omega}\frac{\partial \omega}{\partial y} \quad \overset{(1.7.4)}{\Longrightarrow} \quad \frac{1}{\mu}\frac{d\mu}{d\omega} = \frac{\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}}{Q\dfrac{\partial \omega}{\partial x} - P\dfrac{\partial \omega}{\partial y}} \overset{(1.7.5)}{=} \chi(\omega)$$

Now we need to solve the separable ODE with respect to F:

$$\frac{1}{\mu}\frac{d\mu}{d\omega} = \chi(\omega) \quad \Longrightarrow \quad \ln|\mu| = \int \chi(\omega) d\omega \quad \Longrightarrow \quad \mu(\omega) = \exp\left(\int \chi(\omega) d\omega\right).$$

1.7. ODEs that reduce to exact ODEs. Integrating factors

Note 1: The theorem shows that if there is $\omega(x,y)$ that satisfies Eq. (1.7.5), then integrating factor can be easily found, since the partial differential equation Eq. (1.7.4) reduces to an ODE.

Note 2: If an integrating factor exists, it is non-unique. If $F(x,y)$ is an integrating factor, then $F_1 = H(u(x,y))F(x,y)$ is also an integrating factor (Here $H(u)$ is an arbitrary differentiable function): Let's check that $F_1 P$ and $F_1 Q$ satisfy the criterion of exactness given by Eq. (1.7.3):

$$\frac{\partial(F_1 P)}{\partial y} = \frac{dH}{du}\frac{\partial u}{\partial y}FP + H\frac{\partial(FP)}{\partial y} = \frac{dH}{du}FQ\frac{\partial u}{\partial x} + H\frac{\partial(FQ)}{\partial x} = \frac{\partial(F_1 Q)}{\partial x}.$$

Here we use Eqs. (1.7.2) and (1.7.3) for $F(x,y)$: $\frac{\partial u}{\partial y} = FQ, \quad FP = \frac{\partial u}{\partial x}$

Note 3: Search for integrating factors is usually limited by simple $\omega(x,y)$. Examples:

1. $\omega = x$, then Eq. (1.7.5) reduces to $\chi(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

2. $\omega = y$, then Eq. (1.7.5) reduces to $\chi(y) = -\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

3. $\omega = x + y, \quad \omega = x - y, \quad \omega = xy, \quad \omega = x^2 + y^2$, etc.

Example 1: $(x^2 + y^2 + x)dx + ydy = 0.$

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 2y \quad \Longrightarrow \quad \text{Let's try} \quad \omega = x \quad \Longrightarrow \chi(x) = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 2 \Longrightarrow F = e^{\int 2dx} = e^{2x}$$

$$e^{2x}(x^2 + y^2 + x)dx + e^{2x}ydy = 0: \quad \text{Exact equation,} \qquad \text{solution } (x^2 + y^2)e^{2x} = c.$$

1.7. ODEs that reduce to exact ODEs. Integrating factors

Example 2: Find an integrating factor and solve the equation

$$(x^2 + y^2 + y)dx - xdy = 0$$

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 2y + 1 + 1 = 2(y+1) \implies \text{Let's try } \omega = x^2 + y^2 \implies$$

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q\frac{\partial \omega}{\partial x} - P\frac{\partial \omega}{\partial y}} = \frac{2(y+1)}{-x \cdot (2x) - (x^2 + y^2 + y) \cdot (2y)} = -\frac{1}{x^2 + y^2} = -\frac{1}{\omega} \implies$$

$$\text{Integrating factor exists: } \mu = e^{-\int d\omega/\omega} = \frac{1}{\omega} \qquad F = \frac{1}{\omega(x,y)} = \frac{1}{x^2 + y^2}$$

$$\frac{x^2 + y^2 + y}{x^2 + y^2}dx - \frac{x}{x^2 + y^2}dy = 0: \quad \text{Exact equation}$$

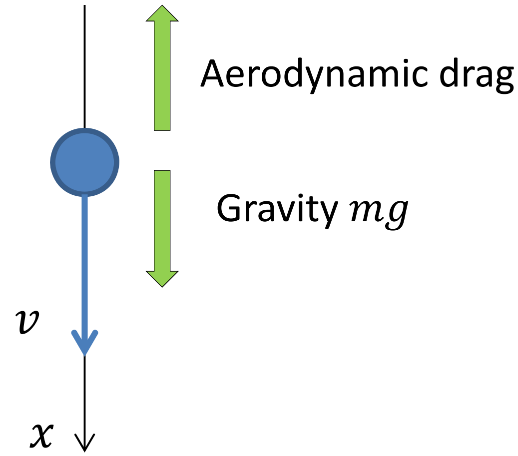
Solution:

$$x + \arctan\frac{x}{y} = c$$

Prompting: Use the rule for the inverse tangent of reciprocal argument:

$$\arctan\frac{1}{x} = \pm\frac{\pi}{2} - \arctan x$$

1.8. Relaxation and equilibrium



- Let's consider a rain droplet: A sphere of radius R and mass m that moves in air along the vertical direction.
- The motion of the sphere is affected by two forces: Gravity and **aerodynamic drag** force.

Equation of motion of the sphere (Newton's 2nd law of motion) can be written as follows

$$m\frac{dv}{dt} = mg - \frac{1}{2}C_d \rho A |v| v \qquad (1.8.1)$$

Drag force always decelerates the body, so its direction is opposite to velocity

where v is velocity of the sphere, g is the gravitational acceleration, C_d is the sphere **drag coefficient**, ρ is the air density, and $A = \pi R^2$ is the sphere cross section. We assume that

$$C_d = C_d(Re)$$

Here $Re = 2R\rho|v|/\mu$ is the **Reynolds number**; μ is the air viscosity. It means that $C_d = C_d(|v|)$.

We assume that in the initial state $t = 0$ the sphere has the initial velocity v_0:

$$\text{At } t = 0: \quad v(0) = v_0 \qquad (1.8.2)$$

Our goal is to predict the droplet velocity by solving the problem given by Eqs. (1.8.1), (1.8.2).

1.8. Relaxation and equilibrium

Preliminary analysis of the problem

$$m\frac{dv}{dt} = mg - \frac{1}{2}C_d(v)\rho A|v|v \qquad (1.8.3)$$

- We can notice that the gravity and drag forces tend to act in the opposite directions. It means that at some $v = v_\infty$ they will *counterbalance* each other.
- The situation, when there are factors that drive the system under consideration in opposite "directions" is quite common in engineering and natural sciences.
- The state of our system when $v = v_\infty$ and $dv/dt = 0$ is called the **equilibrium state**.
- In this problem, the equilibrium state is the state when the sphere moves with constant equilibrium (terminal) velocity v_∞ given by the condition $dv/dt = 0$, i.e.

This is the *algebraic* equation that predicts the equilibrium state

$$\frac{1}{2}C_d(v_\infty)\rho A|v_\infty|v_\infty = mg \qquad (1.8.4)$$

- The equilibrium state is the state that is established at $t \to \infty$.
- Our ODE describes the dynamical process of approaching the equilibrium state from arbitrary initial state. The process of transition to the equilibrium state is called the **relaxation**.
- Eq. (1.8.3) is the simple example of **relaxation equations** describing the relaxation processes.
- Equilibrium state can be established without solving the dynamical equation.

1.8. Relaxation and equilibrium

Let's consider the case when Re << 1. In this case, the sphere C_d is given by the **Stokes equation**

$$C_d = \frac{24}{Re} = \frac{12\mu}{R\rho|v|}$$

This is the result of the accurate solution of the Navier-Stokes equations for fluid flow! Then

$$\frac{dv}{dt} = g - \frac{12A\mu}{Rm}v$$

This is the first-order linear non-homogeneous ODE.

We can notice that the coefficient in the R.H.S. has unit of the inverse time, so let's introduce

$$\tau = \frac{Rm}{12A\mu}$$

Then

$$\frac{dv}{dt} = \frac{g\tau - v}{\tau}$$

The equilibrium velocity v_∞ is given by the condition $dv/dx = 0$, i.e.

$$v_\infty = g\tau$$

The solution of the Cauchy problem with the initial conditions given by Eq. (1.8.2) reads

$$v(t) = v_\infty + (v_0 - v_\infty)e^{-t/\tau} \qquad (1.8.5)$$

This solution describes the relaxation of the sphere velocity from the initial velocity v_0 to the equilibrium velocity v_∞.

1.8. Relaxation and equilibrium

$$v(t) - v_\infty = (v_0 - v_\infty)e^{-t/\tau}$$

$$v(t+\tau) - v_\infty = (v_0 - v_\infty)e^{-1-t/\tau}$$

$$\frac{v(t+\tau) - v_\infty}{v(t) - v_\infty} = \frac{1}{e}$$

Parameter τ is equal to the time which is required to reduce the difference between the current state and equilibrium state in e times. In relaxation problems, this parameter is called the **relaxation time**.

Let's consider the case when Re >> 1. In this case, the sphere drag coefficient $C_d \approx const$, and

$$\frac{dv}{dt} = g - \frac{C_d \rho A}{2m}|v|v = g - \alpha|v|v, \qquad \alpha = \frac{C_d \rho A}{2m} \tag{1.8.6}$$

This is the first-order separable ODE. Let's consider only the case when $v \geq 0$ and $|v|v = v^2$.

The equilibrium velocity in this case is equal to

$$v_\infty = \sqrt{\frac{g}{\alpha}}$$

and Eq. (1.8.6) can be re-written as

$$\frac{dv}{v^2 - v_\infty^2} = -\alpha dt$$

This is an example of non-linear relaxation equation.

1.8. Relaxation and equilibrium

In order to find the integral in the L.H.S, let's re-write it as

$$\frac{1}{v^2 - v_\infty^2} = \frac{1}{(v - v_\infty)(v + v_\infty)} = \frac{A}{v - v_\infty} + \frac{B}{v + v_\infty} = \frac{A(v + v_\infty) + B(v - v_\infty)}{(v - v_\infty)(v + v_\infty)}$$

From the comparison of the left and right-hand sides in this equation we can conclude that

$A + B = 0$ and $A - B = 1/v_\infty$, i.e. $A = 1/(2v_\infty)$ and $B = -1/(2v_\infty)$, i.e.

$$\frac{dv}{v - v_\infty} - \frac{dv}{v + v_\infty} = -2\alpha v_\infty dt$$

Integration results in

$$\log\frac{v - v_\infty}{v_0 - v_\infty} - \log\frac{v + v_\infty}{v_0 + v_\infty} = -2\alpha v_\infty t \qquad \Rightarrow \qquad \frac{v - v_\infty}{v + v_\infty} = \frac{v_0 - v_\infty}{v_0 + v_\infty} e^{-2\alpha v_\infty t}$$

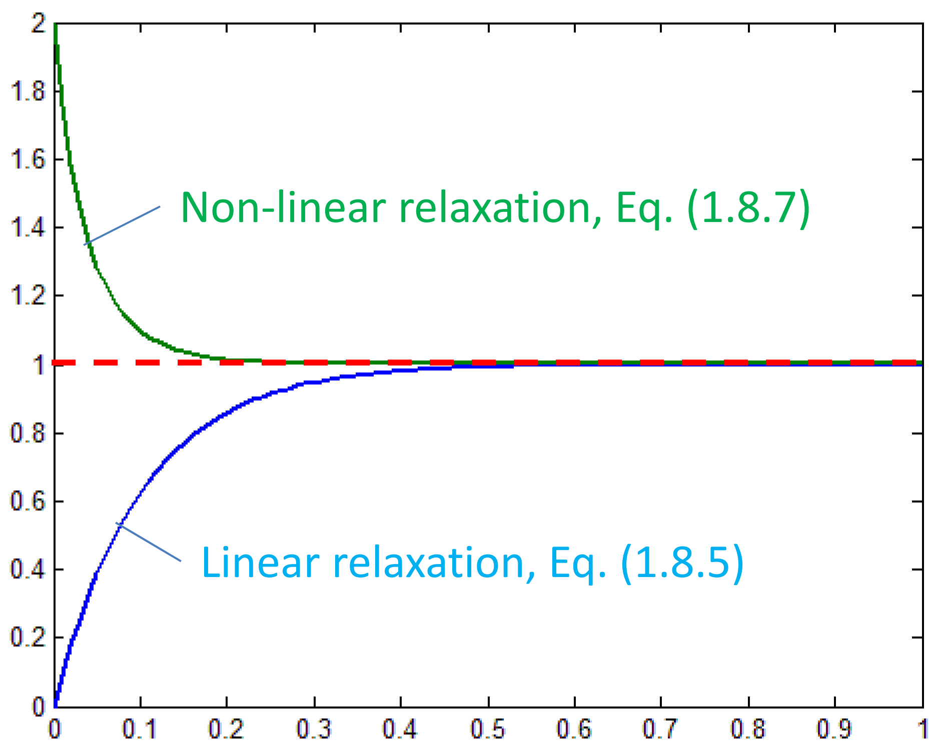
$$v = v_\infty \frac{(v_0 + v_\infty) + (v_0 - v_\infty)e^{-2\alpha v_\infty t}}{(v_0 + v_\infty) - (v_0 - v_\infty)e^{-2\alpha v_\infty t}} \qquad (1.8.7)$$

This equation describes non-linear relaxation of velocity from v_0 to v_∞.

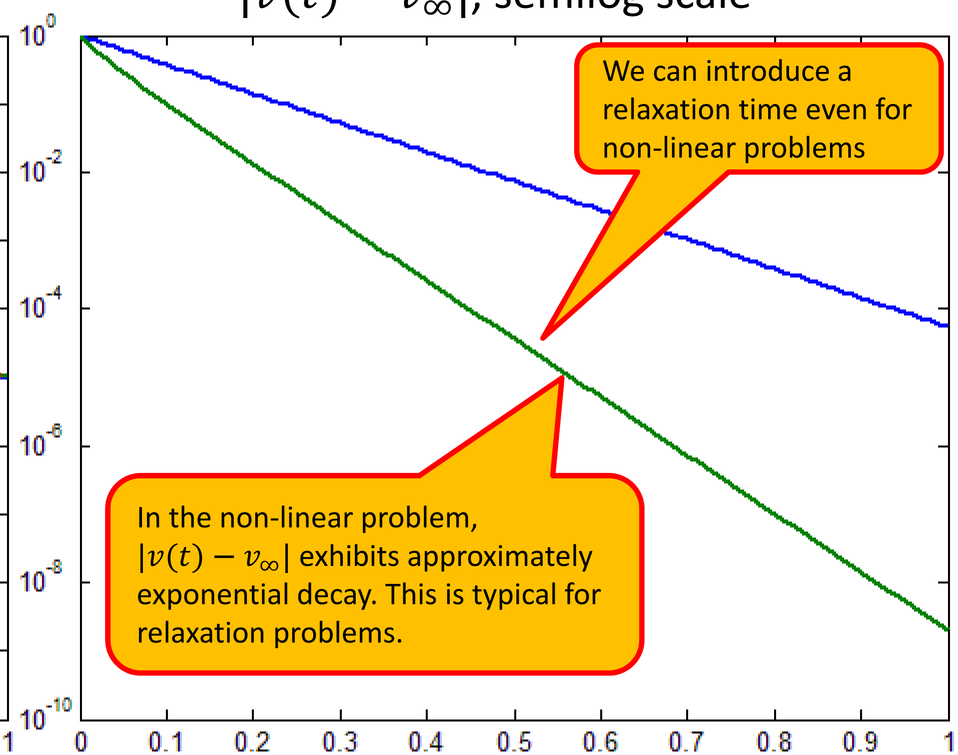
1.8. Relaxation and equilibrium

For simplicity, let's consider the case $\tau = 1/g$, $\alpha = g$, v_∞= 1 m/s, $|v_0 - v_\infty|$ = 1 m/s

Velocity vs. time



$|v(t) - v_\infty|$, semilog scale



$$v(t) = v_\infty + (v_0 - v_\infty)e^{-t/\tau}$$

Relaxation time is τ

$$v = v_\infty \frac{(v_0 + v_\infty) + (v_0 - v_\infty)e^{-2\alpha v_\infty t}}{(v_0 + v_\infty) - (v_0 - v_\infty)e^{-2\alpha v_\infty t}}$$

Relaxation time is $\sim \frac{1}{2\alpha v_\infty}$

1.8. Relaxation and equilibrium

Do we really need to solve the relaxation equation?

- The answer depends on **other time scales** in the considered problem.
- Assume that we are interested in the velocity of a rain droplet after its fall from height H.
- The typical time required for this fall is equal to H/v_∞.
- If $H/v_\infty < \tau$, the process is **non-equilibrium** and in order to describe it we really need to solve the relaxation equation.
- If $H/v_\infty \gg \tau$, the process is **quasi-equilibrium** and small error is made if we replace $v(t)$ with v_∞. In this case, it is enough to find only the equilibrium state, while the solution of full relaxation equation is redundant.