Chapter 2
Second-order ordinary differential equations (ODEs)

Reading:

Kreyszig, *Advanced Engineering Mathematics, 10th Ed.,* 2011

Selection from chapter 2

Prerequisites:

Kreyszig, *Advanced Engineering Mathematics, 10th Ed.,* 2011

- Basics of matrixes and vectors: Section 4.0
- Complex numbers: Sections 13.1, 13.2 and 13.5

Topics for self-studying: 2.4. Euler-Cauchy equations

2.1. Second-order ODEs. Initial and boundary value problems

General form of the **2nd order ODE**

$$f(x, y, y', y'') = 0 \tag{2.1.1}$$

We will consider only equations in the **explicit (normal) form**, resolved with respect to the highest derivative

$$y'' = f(x, y, y') \tag{2.1.2}$$

The general solution of Eq. (2.1.1) or (2.1.2) depends on two arbitrary constants c_1 and c_2 and can be represented in the form

$$G(x, y, c_1, c_2) = 0 \tag{2.1.3}$$

Example: $y'' = f(x)$

$$y'(x) = \int f(x)dx + c_1 = h(x, c_1)$$

$$y(x) = \int \left[\int f(x)dx + c_1 \right] dx + c_2 = g(x, c_1, c_2)$$

Any particular choice of constants c_1 and c_2 gives a particular solution of Eq. (2.1.2).

2.1. Second-order ODEs. Initial and boundary value problems

One possible way to determine c_1 and c_2 is to specify the **initial conditions**

$$y(x_0) = y_0, \qquad y'(x_0) = y_0' \tag{2.1.4}$$

The problem (2.1.2), (2.1.4) is called the **initial value** (or **Cauchy**) **problem** for the 2nd order ODE.

Note: In mechanical problems, if $x = t$ is time and $y(t)$ is coordinate, then $y'(t)$ is velocity and $y''(t)$ is acceleration. It means that in order to solve the 2nd order ODE with respect to the coordinate (Newton's second law of motion) we need to specify the initial position of the body and its initial velocity.

If we know the general solution, Eq. (2.1.3), then in order solve the Cauchy problem (2.1.2), (2.1.4) we can find c_1 and c_2 by solving two equations:

$$G(x, y, c_1, c_2) = 0, \qquad \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} y' = 0$$

Inserting given x_0, y_0, and y_0', one can obtain:

$$G(x_0, y_0, c_1, c_2) = 0, \qquad \frac{\partial G}{\partial x}(x_0, y_0, c_1, c_2) + \frac{\partial G}{\partial y}(x_0, y_0, c_1, c_2) {y_0}' = 0$$

: Two algebraic equations with respect to c_1 and c_2 .

Example: $y'' = f(x)$

$${y_0}' = h(x_0, c_1) \qquad y_0 = g(x_0, c_1, c_2)$$

2.1. Second-order ODEs. Initial and boundary value problems

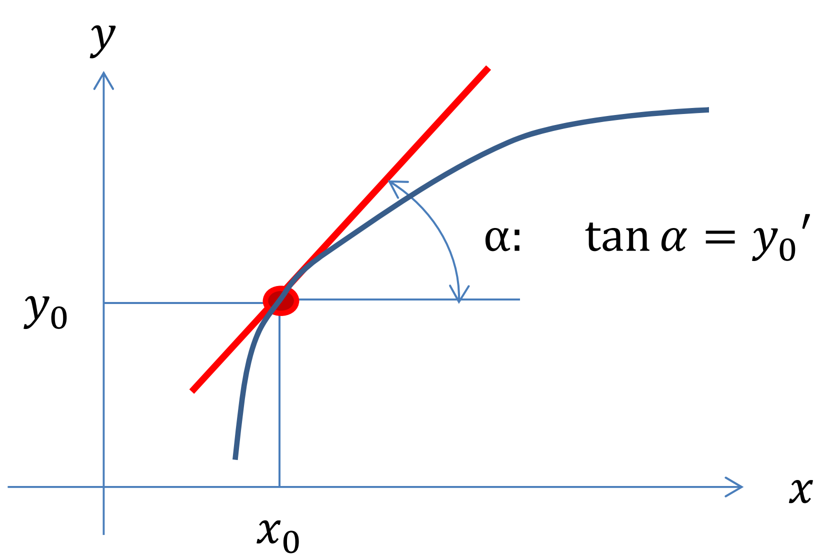
For ODEs of the 2nd and higher orders conditions that allow one to find a particular solution can be specified not only in the form of the initial conditions, but also in other forms.

For example, for ODE (2.1.2) such conditions can be specified in the form of **boundary conditions**

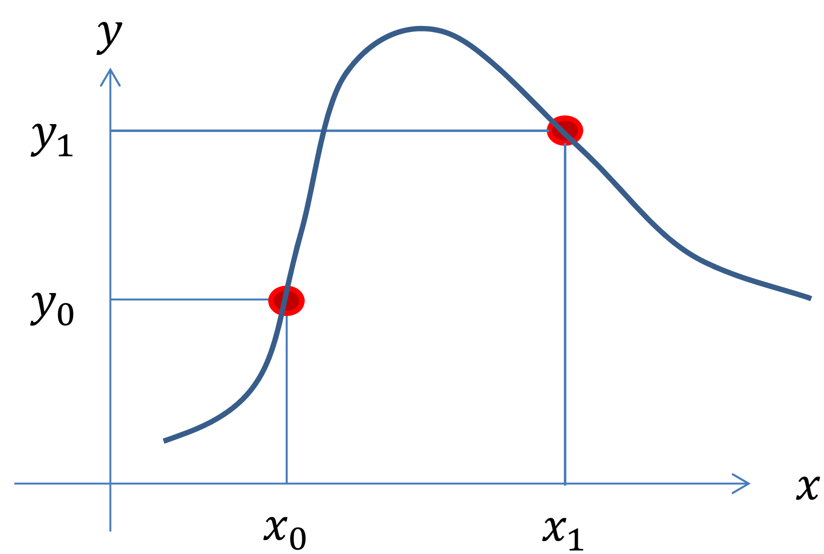
$$y(x_0) = y_0, \qquad y(x_1) = y_1 \tag{2.1.5}$$

The problem (2.1.2), (2.1.5) is called the **boundary value problem**. It is formulated as follows: To find a particular solution of Eq. (2.1.2) that exists al least in the interval $[x_0, x_1]$ and satisfies conditions (2.1.5) at the boundaries of this interval. Another formulation: To find an integral curve of Eq. (2.1.2), which goes through points (x_0, y_0) and (x_1, y_1) on the plane (x, y).

Initial value problem



Boundary value problem



2.1. Second-order ODEs. Initial and boundary value problems

If we know the general solution of Eq. (2.1.2), then the solution of the boundary value problem reduces to the choice of the appropriate constants c_1 and c_2. If we know the general solution in the form of Eq. (2.1.3) then c_1 and c_2 can be found as roots of two algebraic equation:

$$G(x_0, y_0, c_1, c_2) = 0, \qquad G(x_1, y_1, c_1, c_2) = 0$$

: Two algebraic equations with respect to c_1 and c_2 .

Note 1: Boundary conditions can be specified not only in the form of prescribed values of the unknown function, but also in the form of the prescribed value of the derivative, or even in a more complex form:

1. $y(x_0) = y_0$: **Dirichlet boundary condition**
2. $y'(x_0) = {y_0}'$: **Neumann boundary condition**
3. $ay(x_0) + by'(x_0) = c$: **Robin boundary condition** (a, b, and c are arbitrary constants)

Note 2: For given boundary conditions, the solution of the boundary value problem may/may not exist, and, if exists, it can be unique or not. Conditions that ensure existence and uniqueness of the boundary value problems are formulated in the form of the **existence and uniqueness theorems**.

Note 3: Initial and boundary value problems are general for many engineering problems. Boundary value problems naturally appear if the want to know which initial state allows us to reach the desired final state .

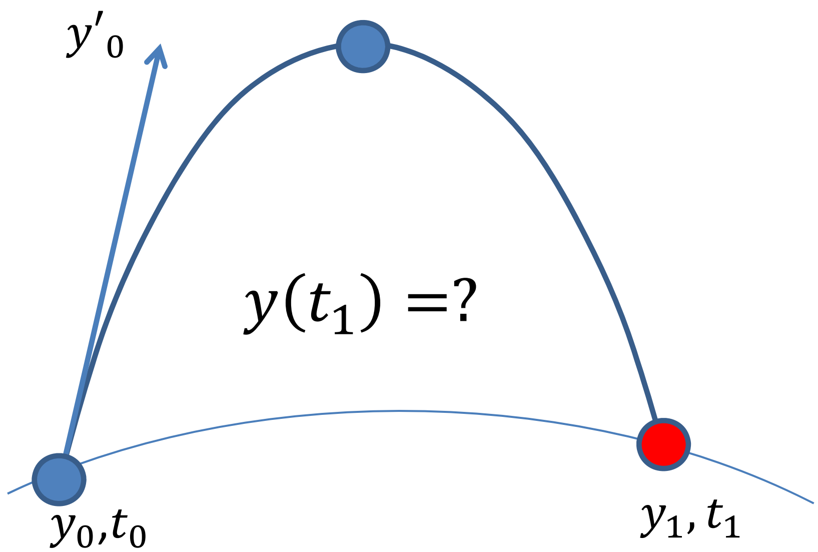
2.1. Second-order ODEs. Initial and boundary value problems

Example 1: Motion of a body (bullet).

We solve second-order ODEs which represent Newton's second law of motion.

Initial value problem:

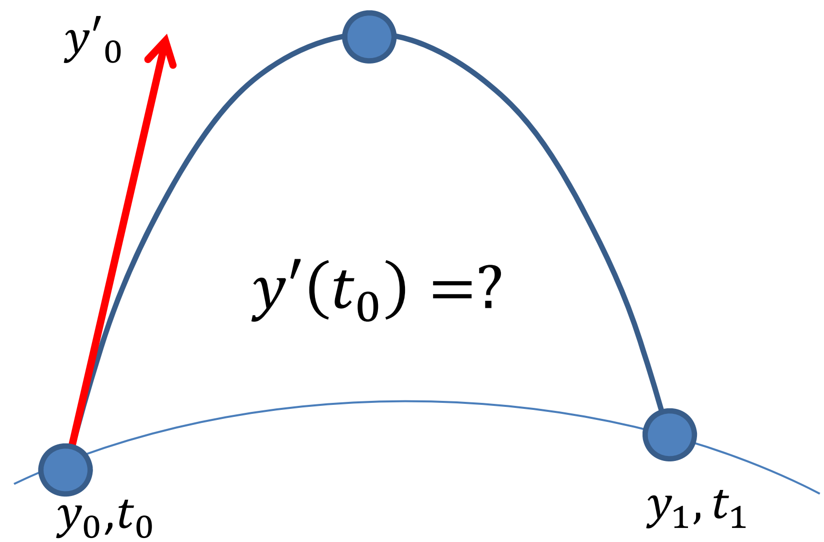
We fix the initial position of the body y_0 and its velocity $y_0{}'$ at time $t = t_0$ and want to know the position of the body at time $t = t_1$



Where the final position of the missile?

Boundary value problem:

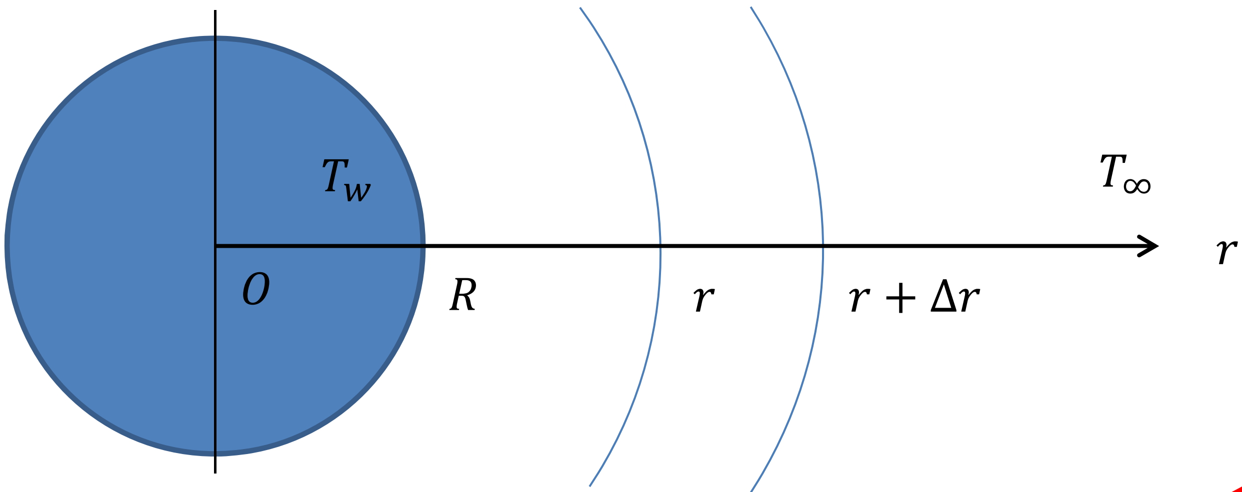
We fix the initial position of the body y_0 at time $t = t_0$ and want to know which velocity $y_0{}'$ allows the bullet to hit the target in given position y_1 at time $t = t_1$



How to hit the target?

2.1. Second-order ODEs. Initial and boundary value problems (optional)

Example 2: Steady-state heat transfer of a sphere in a quiescent fluid.



T_w and T_∞ are boundary conditions for the ODE with respect to fluid temperature

1. The sphere is isothermal and has temperature T_w
2. The fluid has temperature T_∞ infinitely far from the sphere.
3. Heat conduction in the fluid is described by the Fourier law, the fluid thermal conductivity, k, is constant

$$Q(r) = -4\pi r^2 k \frac{dT}{dr}$$

$Q(r)$ is the **heat flux**, i.e. the amount of energy transferred through a spherical surface of area $4\pi r^2$ per unit time

4. Energy conservation for the steady-state process

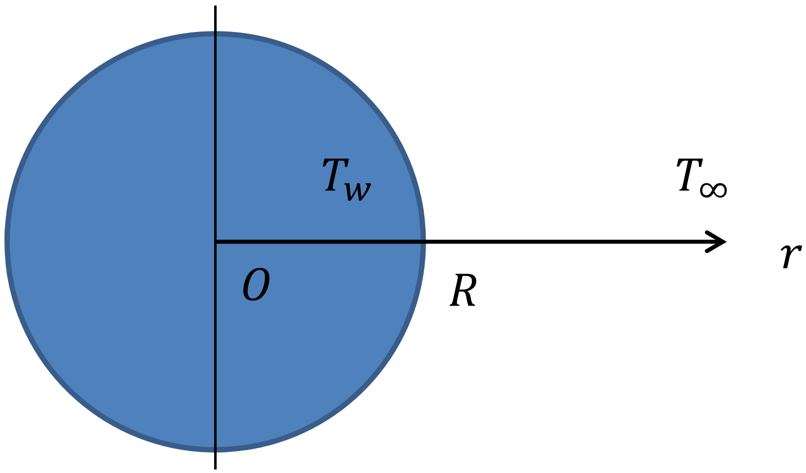
$$\Delta U(r, r + \Delta r) = [Q(r) - Q(r + \Delta r)]\Delta t = 0$$

$$\frac{Q(r + \Delta r) - Q(r)}{\Delta r} \xrightarrow[\Delta r \to 0]{} \frac{d}{dr}\left(4\pi r^2 k \frac{dT}{dr}\right) = 0$$

$\Delta U(r, r + \Delta r)$ is the increment of energy in the spherical layer between r and $r + \Delta r$ during time Δt

2.1. Second-order ODEs. Initial and boundary value problems

Example 2: Steady-state heat transfer of a sphere in a quiescent fluid.



$$\frac{Q(R)}{4\pi R^2} = -k\left.\frac{dT}{dr}\right|_{r=R} = h(T_w - T_\infty)$$

Newton's law of cooling

h: **Heat transfer coefficient**

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$$: Steady state, spherically-symmetric **heat conduction equation**

$$T(R) = T_w$$: Boundary condition at the sphere surface

$$T(r) \xrightarrow[r\to\infty]{} T_\infty$$: Boundary condition far from the sphere

Here we solve a problem with the Dirichlet boundary conditions

Solution: $r^2 T' = c_1 \quad \Rightarrow \quad T' = \dfrac{c_1}{r^2} \quad \Rightarrow \quad T(r) = -\dfrac{c_1}{r} + c_2$

$$c_2 = T_\infty$$
$$c_1 = (T_\infty - T_w)R$$

$$\boxed{T(r) = T_\infty + (T_w - T_\infty)\frac{R}{r}}$$

$$\left.\frac{dT}{dr}\right|_{r=R} = \frac{T_\infty - T_w}{R} \qquad h = \frac{k}{R}$$

Nusselt number: $\boxed{Nu = 2Rh/k = 2}$

2.1. Second-order ODEs. Initial and boundary value problems

Example 2: Steady-state heat transfer of a sphere in a quiescent fluid.

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0 \qquad \text{General solution: } T(r) = -\frac{c_1}{r} + c_2 \qquad T' = \frac{c_1}{r^2}$$

$$T(r) \xrightarrow[r\to\infty]{} T_\infty \qquad \Longrightarrow \qquad c_2 = T_\infty$$

Let's use other boundary conditions at the sphere surface:

1. Neumann condition:

$$-k\left.\frac{dT}{dr}\right|_{r=R} = q_w \Longrightarrow c_1 = -\frac{q_w R^2}{k}: \qquad T(r) = T_\infty + \frac{q_w R^2}{kr}$$

2. Robin condition:

$$-k\left.\frac{dT}{dr}\right|_{r=R} = h(T(R) - T_\infty) \quad \Longrightarrow \quad -\frac{k\cancel{c_1}}{R^2} = h\left(\cancel{T_\infty} - \frac{\cancel{c_1}}{R} - \cancel{T_\infty}\right)$$

The solution exists only if $h = k/R$, but such a solution is not unique, since arbitrary constant c_1 (or surface temperature T_w) satisfies all boundary conditions.

2.1. Second-order ODEs. Initial and boundary value problems

We will consider only one general type of second-order ODEs, namely, the **linear** 2nd order ODEs that have the following form

$$f_2(x)y'' + f_1(x)y' + f_0(x)y = h(x) \qquad (2.1.6)$$

We will assume that in the interval $I = (a, b)$, i.e. $a < x < b$, where we look for the solution of Eq. (2.1.6), $f_2(x) \neq 0$, so that Eq. (2.1.6) can be re-written in the **standard form** (coefficient at y'' is equal to 1):

$$y'' + p(x)y' + q(x)y = r(x) \qquad (2.1.7)$$

Linear ODE (2.1.7) is called **homogeneous** if $r(x) = 0$, otherwise it is called **nonhomogeneous**.

2.2. Second-order linear homogeneous ODEs

Prerequisites: Solution of a linear system

System of two linear equations:
$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Matrix of coefficients $\mathbf{M}$, RHS vector $\mathbf{b}$, and vector of unknowns $\mathbf{x}$:

$$\mathbf{M} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad : \qquad \mathbf{Mx} = \mathbf{b}$$

Determinant:

$$\det \mathbf{M} = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Inverse matrix: $\mathbf{M}^{-1}$ is inverse to $\mathbf{M}$ if $\mathbf{M}^{-1}\mathbf{M} = \mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\text{if} \det \mathbf{M} \neq 0 \text{ then} \qquad \mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Solution of the linear system:

$$\text{if} \det \mathbf{M} \neq 0 \text{ then solution exists and unique:} \qquad \mathbf{x} = \mathbf{M}^{-1}\mathbf{b} = \frac{1}{\det \mathbf{M}} \begin{pmatrix} a_{22}b_1 - a_{12}b_2 \\ -a_{21}b_1 + a_{11}b_2 \end{pmatrix}$$

For the homogeneous system ($b_1 = b_2 = 0$, $\mathbf{b} = 0$):

If $\det \mathbf{M} \neq 0$, then only **trivial solution** $\mathbf{x} = 0$ exists.

If $\det \mathbf{M} = 0$, then multiple solutions exist.

2.2. Second-order linear homogeneous ODEs

Our goal is to study the general properties of solution of the linear homogeneous equation of the 2nd order

$$y'' + p(x)y' + q(x)y = 0 \qquad (2.2.1)$$

Functions $p(x)$ and $q(x)$ are called **coefficients** of Eq. (2.2.1).

Existence and uniqueness theorem:

If $p(x)$ and $q(x)$ are continuous functions in some open interval $I = (a, b)$, then the initial value problem for Eq. (2.2.1) with initial conditions

$$y(x_0) = y_0, \qquad y'(x_0) = y'_0 \qquad (2.2.2)$$

where $a < x_0 < b$, has a unique solution in interval (a, b).

We will not prove this theorem, but our goal is to use this theorem in order to study the general form of solutions of Eq. (2.2.1).

Linear combination of functions $y_1(x)$ and $y_2(x)$ is a function $y(x) = c_1 y_1(x) + c_2 y_2(x)$ where c_1 and c_2 are two arbitrary constants.

Theorem: **Superposition principle** for solutions of linear homogeneous second-order ODEs:

If $y_1(x)$ and $y_2(x)$ are two particular solutions of Eq. (2.2.1), then any their linear combination, $c_1 y_1(x) + c_2 y_2(x)$, is also a solution.

2.2. Second-order linear homogeneous ODEs

Proof: $$(c_1y_1 + c_2y_2)''+p(x)(c_1y_1 + c_2y_2)'+q(x)(c_1y_1 + c_2y_2)$$
$$= c_1[y_1'' + py_1' + qy_1] + c_2[y_2'' + py_2' + qy_2] = 0$$

Note: Superposition principle shows that the general solution of Eq. (2.2.1) includes two arbitrary constants. The problem is that $y(x) = c_1y_1(x) + c_2y_2(x)$ may not be the general solution. Let's consider $y_2(x) = dy_1(x)$. Then $y(x) = c_1y_1(x) + c_2y_2(x) = (c_1 + dc_2)y_1(x)$ contains only one arbitrary constant, so two conditions of the Cauchy problem cannot be satisfied simultaneously.

Let's consider a condition that guaranties that $y(x) = c_1y_1(x) + c_2y_2(x)$ is the general solution of Eq. (2.2.1). We need a few new definitions.

Two functions $y_1(x)$ and $y_2(x)$ are called **linearly independent** on an open interval $I = (a, b)$ if their linear combination is equal to 0 in any point x of this interval only if $c_1 = c_2 = 0$, i.e.

$$c_1y_1(x) + c_2y_2(x) = 0 \qquad \text{for all } x \in (a, b) \qquad \Longrightarrow \qquad c_1 = c_2 = 0$$

Otherwise, if there are constant $c_1 \neq 0$ and/or $c_2 \neq 0$ for which $c_1y_1(x) + c_2y_2(x) = 0$ in any point of (a, b), functions $y_1(x)$ and $y_2(x)$ are called **linearly dependent** on (a, b).

Note: Linearly dependent functions are proportional to each other in (a, b):

If $c_1 \neq 0 \quad \Longrightarrow \quad y_1(x) = -(c_2/c_1)y_2(x)$.

If $c_2 \neq 0 \quad \Longrightarrow \quad y_2(x) = -(c_1/c_2)y_1(x)$.

Example: $y_1(x) = 2\cos x$ and $y_2(x) = 5\sin(\frac{\pi}{2} - x)$ are linearly dependent; $y_1(x) = x$ and $y_1(x) = x^2$ are linearly independent.

2.2. Second-order linear homogeneous ODEs

If functions $y_1(x)$ and $y_2(x)$ have derivatives $y_1{}'(x)$ and $y_2'(x)$ then the determinant

$$W(y_1, y_2, x) = W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = y_1 y_2' - y_2 y_1{}' \qquad (2.2.3)$$

is called the **Wronskian** of these functions in point x. *Wronskian = sign of linear dependency*.

Theorem 1:

If $y_1(x)$ and $y_2(x)$ are linearly dependent on (a, b) then the Wronskian of these functions is 0 in any point $x \in (a, b)$.

Proof: If y_1 and y_2 are linearly dependent, they are proportional to each other. If, e.g., $c_1 \neq 0$, $y_1 = -(c_2/c_1)y_2$, $y_1{}' = -(c_2/c_1)y_2{}'$, then $W(x) = -(c_2/c_1)y_2 y_2{}' + (c_2/c_1)y_2 y_2{}' = 0$.

Theorem 2:

If coefficients $p(x)$ and $q(x)$ in Eq. (2.2.1) are continuous functions in some open interval (a, b) and for two particular solutions of this equation $y_1(x)$ and $y_2(x)$ there is a $x_0 \in (a, b)$ such that $W(x_0) = 0$, then $y_1(x)$ and $y_2(x)$ are linearly dependent solutions in (a, b).

Proof: [We need to find non-zero k_1 and k_2: $k_1 y_1(x) + k_2 y_2(x) = 0$ for any $x \in (a, b)$]

Let's start from point x_0 and consider a linear system with respect to k_1 and k_2:

$$\begin{gathered} k_1 y_1(x_0) + k_2 y_2(x_0) = 0 \\ k_1 y_1{}'(x_0) + k_2 y_2{}'(x_0) = 0 \end{gathered} \qquad (2.2.4)$$

The determinant of the matrix of coefficients of this system is $W(x_0)$ and is equal to 0.

2.2. Second-order linear homogeneous ODEs

Thus, we have a homogeneous linear system with zero determinant. This system has a non-trivial solution $k_1 \neq 0$ and/or $k_2 \neq 0$. Using this solution, let's introduce a new function

$$y(x) = k_1 y_1(x) + k_2 y_2(x)$$

According to the superposition principle, $y(x)$ is also a solution in Eq. (2.2.1). From Eq. (2.2.4) it is obvious that $y(x)$ satisfies initial conditions $y(x_0) = 0$, $y'(x_0) = 0$. But the trivial solution $y^*(x) = 0$ also satisfies these conditions. Since for continuous $p(x)$ and $q(x)$ the **solution is unique**, $y(x) = k_1 y_1(x) + k_2 y_2(x) = y^*(x) = 0$, i.e. $y_1(x)$ and $y_2(x)$ are linearly dependent.

Theorem 3:

If coefficients $p(x)$ and $q(x)$ in Eq. (2.2.1) are continuous functions in some open interval (a, b) and $y_1(x)$ and $y_2(x)$ are two particular solutions of this equation , then

1. If there is a $x_0 \in (a, b)$ such that $W(x_0) = 0$, then $W(x) = 0$ for any $x \in (a, b)$.
2. If there is a $x \in (a, b)$ such that $W(x) \neq 0$, then $y_1(x)$ and $y_2(x)$ are linearly independent in (a, b) .
3. If $y_1(x)$ and $y_2(x)$ are linearly independent on (a, b) , then $W(x) \neq 0$ in (a, b).

Proof:

1. If $W(x_0) = 0$ then $y_1(x)$ and $y_2(x)$ are linearly dependent in (a, b) (Theorem 2) $\Longrightarrow$ $W(x) = 0$ for any $x \in (a, b)$ (Theorem 1).
2. If we assume that $y_1(x)$ and $y_2(x)$ are linearly dependent, then $W(x) = 0$ for any $x \in (a, b)$ (Theorem 1). But it contradicts to the condition in statement 2, so $y_1(x)$ and $y_2(x)$ should be linearly independent.

2.2. Second-order linear homogeneous ODEs

3. If we assume that there is a $x_0 \in (a, b)$ such that $W(x_0) = 0$, then $y_1(x)$ and $y_2(x)$ should be linearly dependent according to Theorem 2.

Theorem 4: **Structure of the general solution of a homogeneous linear ODE**

If coefficients $p(x)$ and $q(x)$ in Eq. (2.2.1) are continuous functions in some open interval (a, b) then

1. Eq. (2.2.1) has a general solution in the form

$$y(x) = \mathrm{c}_1 y_1(x) + \mathrm{c}_2 y_2(x) \qquad (2.2.5)$$

   where and $y_1(x)$ and $y_2(x)$ are two *linearly independent* particular solutions.
2. The general solution (2.2.5) includes all solutions, i.e. (2.2.1) has no singular solutions, and any solution can be represented in the form (2.2.5).

Proof:

1. According to existence and uniqueness theorem, there are two unique solutions $y_1(x)$ and $y_2(x)$ that satisfy the initial conditions

$$y_1(x_0) = 1, {y_1}'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0, {y_2}'(x_0) = 1.$$

   For these solutions, $W(x_0) = 1 \neq 0$ and, thus, they are linearly independent according to Theorem 3 (statement 2). From the superposition principle, Eq. (1.9.5) with arbitrary c_1 and c_2 is also a solution, thus it is the general solution.
2. Now let's proof that solution of the Cauchy problem with arbitrary initial conditions

$$y(x_0) = y_0, \qquad y'(x_0) = y'_0$$

can be represented in the form of Eq. (2.2.5). Such a solution should satisfy equations:

$$c_1 y_1(x_0) + c_2 y_2(x_0) = y_0$$
$$c_1 y_1{}'(x_0) + c_2 y_2{}'(x_0) = y_0{}'$$

But since $y_1(x)$ and $y_2(x)$ are linearly independent, $W(x_0) \neq 0$, and this linear system has a unique solution. According to the existence and uniqueness theorem, the solution which we found is unique. Thus, any solution can be represented in the form (2.2.5) with proper choice of c_1 and c_2.

Note 1: A pair of linearly independent solutions of Eq. (2.2.1) is called **basis**.

Note 2: There is no general methods to solve Eq. (2.2.1) with arbitrary $p(x)$ and $q(x)$. We will show that the general solution can be found if we know one particular solution of this equation.

Theorem: **Abel's formula**

The Wronskian of two arbitrary solutions of Eq. (2.2.1) can be calculated as

$$\boxed{W(x) = Ce^{-\int p(x)dx}} \quad \text{: Abel's formula} \qquad (2.2.6)$$

where C is a constant.

Proof: If $y_1(x)$ and $y_2(x)$ are two solutions, then

$$y_2'' + p(x)y_2' + q(x)y_2 = 0 \quad | \times y_1$$
$$y_1'' + p(x)y_1' + q(x)y_1 = 0 \quad | \times y_2$$

Subtract Eqs. from each other:

$$y_2''y_1 - y_1''y_2 = -py_2'y_1 \cancel{-qy_1 y_2} + py_1' y_2 \cancel{+qy_2 y_1} = -p(y_2'y_1 - y_1'y_2) = -pW$$

2.2. Second-order linear homogeneous ODEs

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = y_1 y_2' - y_2 y_1{}'$$

$$W' = \cancel{y_1' y_2'} + y_1 \; y_2'' - \cancel{y_2' y_1'} - y_2 \; y_1'' = -pW$$

$$\frac{dW}{W} = -p(x)dx \quad \Longrightarrow \quad W(x) = Ce^{-\int p(x)dx}$$

Liouville theorem:

If we know one solution $y_1(x)$ of Eq. (2.2.1), then another linearly independent solution $y_2(x)$ can be calculated as

$$\boxed{y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1^2(x)} dx} \qquad (2.2.7)$$

Proof:

$$W(x) = y_1 y_2' - y_2 y_1' \quad \Longrightarrow \left(\frac{y_2}{y_1}\right)' = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = \frac{W}{y_1^2}$$

$$\left(\frac{y_2}{y_1}\right)' \overset{\text{Abel's formula}}{=} \frac{e^{-\int p(x)dx}}{y_1^2}$$

Integration of the last Eq. gives us Eq. (2.2.7).

Note: The Liouville theorem allows us to find the general solution of Eq. (2.2.1) if we know one arbitrary solution.

1. First, we need to find the second linearly independent solution from Eq. (2.2.7).
2. Second, the general solution is given by Eq. (2.2.5).

2.2. Second-order linear homogeneous ODEs

Example: $xy'' + 2y' + xy = 0$, $y_1(x) = \cos x \,/x$.

1. Transform Eq. to the standard form:

$$y'' + \left(\frac{2}{x}\right)y' + y = 0: \qquad p(x) = \frac{2}{x} \qquad q(x) = 1$$

2. Apply Liouville theorem: $\quad W = e^{-\int p(x)dx} = e^{-2\ln x} = \dfrac{1}{x^2}$

$$y_2(x) = y_1(x)\int \frac{e^{-\int p(x)dx}}{y_1^2(x)}dx = \frac{\cos x}{x}\int \frac{(1/x)^2 dx}{(\cos x\,/x)^2} = \frac{\cos x}{x}\int \frac{dx}{(\cos x)^2} = \frac{\cos x}{x}\tan x = \frac{\sin x}{x}$$

3. Apply theorem 4:

General solution is $\quad y(x) = \dfrac{c_1 \cos x + c_2 \sin x}{x}$

2.3. Second-order linear homogeneous ODEs with constant coefficients

Prerequisites: Complex numbers

$z = x + iy = (x, y)$: **Complex number**

$x = \mathrm{Re}\, z$: Real part

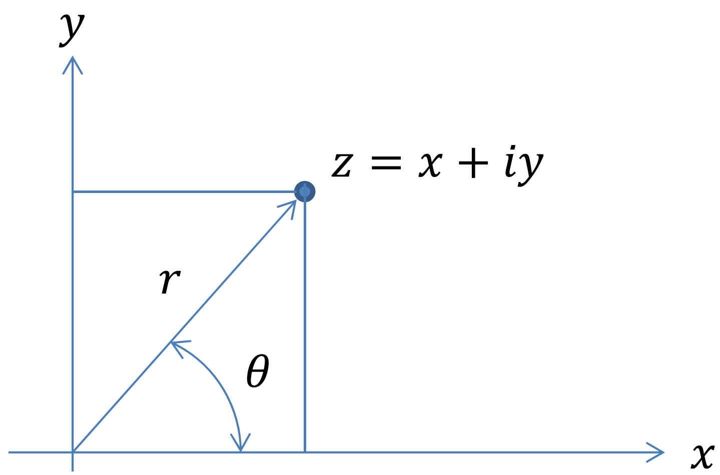
$y = \mathrm{Im}\, z$: Imaginary part

$i = \sqrt{-1} = (0,1)$: Imaginary unit

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + iy_1 x_2 + iy_2 x_1 + i^2 y_1 y_2$$
$$= (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + y_2 x_1)$$

$\bar{z} = x - iy$ is the **complex conjugate** of $z = x + iy$



Polar form of a complex number:

$r = |z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$ is **modulus** (absolute value),

$\tan\theta = y/x$, θ is **argument**

$z = r(\cos\theta + i\sin\theta)$

Complex exponential function:

$e^z = e^{x+iy} = e^x(\cos y + i \sin y\,)$

2.3. Second-order linear homogeneous ODEs with constant coefficients

Second-order linear homogeneous ODE **with constant coefficients** has the standard form

$$y'' + ay' + by = 0 \qquad (2.3.1)$$

where a and b are arbitrary constants. Its general solution has the form

$$y(x) = \mathrm{c}_1 y_1(x) + \mathrm{c}_2 y_2(x) \qquad (2.3.2)$$

where $y_1(x)$ and $y_2(x)$ are two arbitrary linearly independent particular solutions of Eq. (2.3.1). Let's look for a particular solution in the form of the exponential function $y(x) = e^{\lambda x}$ and substitute this function into Eq. (2.3.1). Then

$$\lambda^2 \cancel{e^{\lambda x}} + a\lambda \cancel{e^{\lambda x}} + b\cancel{e^{\lambda x}} = 0 \text{ or}$$
$$\lambda^2 + a\lambda + b = 0 \qquad (2.3.3)$$

Eq. (2.3.3) is called the **characteristic equation** for Eq. (2.3.1). The characteristic equation says us that $y(x) = e^{\lambda x}$ can be a solution only for some particular λ that should satisfy (2.3.3).

Characteristic equation is the quadratic equation and its roots are equal to

$$\lambda_1 = -\frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b} \qquad \lambda_2 = -\frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b} \qquad (2.3.4)$$

2.3. Second-order linear homogeneous ODEs with constant coefficients

Three cases are possible

Case 1: $a^2 > 4b$, Two distinct real roots, $\lambda_1 \neq \lambda_2$

Particular solutions take the form $y_1(x) = e^{\lambda_1 x}$ and $y_2(x) = e^{\lambda_2 x}$. These solutions are linearly independent since

$$W(x) = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} \end{vmatrix} = (\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2) x} \neq 0$$

The general solution takes the form

$$\boxed{y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}} \qquad (2.3.5)$$

Case 2: $a^2 = 4b$, Double real root $\lambda = \lambda_1 = \lambda_2 = -a/2$

One particular solution is $y_1(x) = e^{\lambda x}$. Another linearly independent particular solution can be found with the Liouville theorem:

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x) dx}}{y_1^2(x)} dx = e^{\lambda x} \int \frac{e^{-\int a dx}}{e^{2\lambda x}} dx = e^{\lambda x} \int e^{-(2\lambda + a) x} dx = x e^{\lambda x}$$

Then the general solution takes the form

$$\boxed{y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x} = (c_1 + c_2 x) e^{\lambda x}} \qquad (2.3.6)$$

2.3. Second-order linear homogeneous ODEs with constant coefficients

Case 3: $a^2 < 4b$, Two distinct complex roots

$$\lambda_1 = -\frac{a}{2} + i\omega \qquad \lambda_2 = -\frac{a}{2} - i\omega \qquad \omega = \sqrt{b - \left(\frac{a}{2}\right)^2}$$

Particular solutions take the form $y_1(x) = e^{\lambda_1 x}$ and $y_2(x) = e^{\lambda_2 x}$, but in this case they are *complex-valued exponential functions*. The general solution takes the form

$$y(x) = \mathrm{c}_1 e^{\lambda_1 x} + \mathrm{c}_2 e^{\lambda_2 x}$$

This equation gives us a *real-valued* solution only if c_1 and c_2 are *complex* numbers as well.

Let's reduce the general solution to a form, containing *only real numbers*:

Trigonometric representation of the exponential function: $e^{\alpha + i\beta} = e^{\alpha}(\cos\beta + i\sin\beta\,)$

Then
$$y(x) = e^{-\left(\frac{a}{2}\right)x}[(\mathrm{c}_1 + \mathrm{c}_2)\cos\omega x + i(\mathrm{c}_1 - \mathrm{c}_2)\sin\omega x]$$

If we introduce $c_2 = (A + iB)/2$, $c_1 = (A - iB)/2$, then $\mathrm{c}_1 + \mathrm{c}_2 = A$, $i(\mathrm{c}_1 - \mathrm{c}_2) = -i^2 B = B$.
So the general solution takes the form

$$\boxed{y(x) = e^{-\left(\frac{a}{2}\right)x}[A\cos\omega x + B\sin\omega x]} \qquad (2.3.7)$$

Alternative form of the general solution:

Let's introduce

$$C = \sqrt{A^2 + B^2}, \qquad \tan\varphi = \frac{B}{A} \quad \Longrightarrow \quad A = C\cos\varphi\,, \qquad B = C\sin\varphi$$

2.3. Second-order linear homogeneous ODEs with constant coefficients

$$y(x) = e^{-\left(\frac{a}{2}\right)x}[C\cos\varphi\cos\omega x + C\sin\varphi\ \sin\omega x]$$

$$\boxed{y(x) = Ce^{-\left(\frac{a}{2}\right)x}\cos(\omega x - \varphi)} \quad \text{or} \quad y(x) = Ce^{-\left(\frac{a}{2}\right)x}\sin(\omega x + \delta) \qquad (2.3.8)$$

where $\tan\delta = A/B$. If $x = t$ is the time, then ω and φ is the **angular frequency** and **phase**.

Examples: $y'' + 2y' + 5y = 0$

$a^2 = 4 < 4b = 20$, This is Case 3, $a/2 = 1$, $\omega = \sqrt{b - \left(\frac{a}{2}\right)^2} = 2$

General solution: $y(x) = Ce^{-x}\cos(2x - \varphi)$

Let's find a solution of the Cauchy problem $y(0) = 0$, $y'(0) = 1$

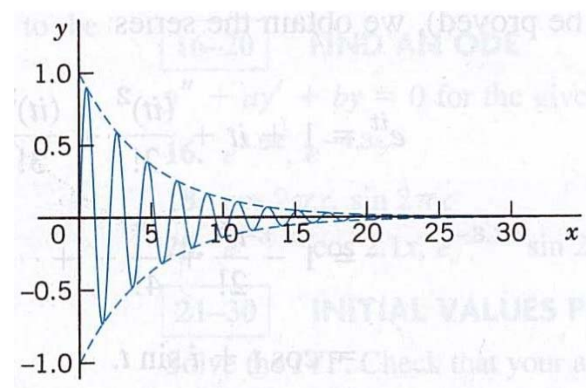
$$y'(x) = -Ce^{-x}\cos(2x - \varphi) - 2Ce^{-x}\sin\ (2x - \varphi)$$
$$y(0) = C\cos(-\varphi) = 0 \quad \Longrightarrow \quad \varphi = \pi/2$$
$$y'(0) = -C(\cos(-\varphi) - 2\sin(-\varphi)) = 1 \Longrightarrow C = -1/2$$

$$y(x) = -\frac{1}{2}e^{-x}\cos\left(2x - \frac{\pi}{2}\right) = \frac{1}{2}e^{-x}\sin 2x$$

$$y'' + 0.4y' + 9.04y = 0$$

$$y(0) = 0,\ y'(0) = 3$$

$$y(x) = e^{-0.2x}\sin 3x$$



Exponential decay + oscillation

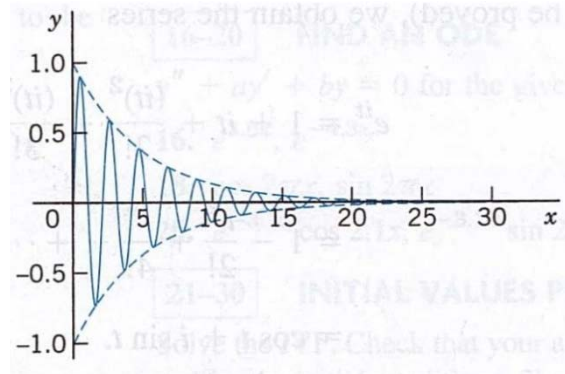
Physical meaning of roots λ_i of the characteristic equations if $x = t$ is the time

Arguments of exponential and trigonometric functions has no physical units. Then τ_R and τ must have units of time.

Two distinct complex root

$$\lambda_{1,2} = -\frac{1}{\tau_R} \pm i\frac{2\pi}{\tau}$$

(2.3.8) $\Rightarrow$ $$y(t) = Ce^{-\frac{t}{\tau_R}} \cos\left(2\pi\frac{t}{\tau} - \varphi\right)$$



$\mathrm{Im}\,\lambda_i = \omega = \frac{2\pi}{\tau}$ defines τ, **period of oscillation** (time between two neighbor maxima).

Re $\lambda_i = -1 / \tau_R$ defines τ_R, time during which the magnitude changes in e times. If $\tau_R > 0$, it is the **relaxation time**.

Two distinct real roots

$$\lambda_i = -\frac{1}{\tau_{Ri}}$$

(2.3.5) $\Rightarrow$ $$y(t) = c_1 e^{-\frac{t}{\tau_{R1}}} + c_2 e^{-\frac{t}{\tau_{R2}}}$$

If $\tau_{Ri} > 0$, then the solution is a combination of two decays with relaxation times τ_{R1} and τ_{R2}.

- **The roots λ_i define two time scales** of the dynamical process described by the ODE.
- **If x is position, then λ_i define two length scales**: **Wave length** and **relaxation length** or two relaxation lengths.

2.4. Euler-Cauchy equations

The **Euler-Cauchy equation** is the second-order linear homogeneous equation of the form

$$x^2y'' + axy' + by = 0 \tag{2.4.1}$$

where a and b are arbitrary constants.

Euler-Cauchy equation is another example of the linear homogeneous ODE of the second-order that can be solved algebraically. Let's try a particular solution in the form $y(x) = x^m$. Inserting such function into (2.4.1) one can obtain:

$$m(m-1)\cancel{x^{m-2+2}} + am\cancel{x^{m-1+1}} + b\cancel{x^m} = 0$$

Thus, we see that $y(x) = x^m$ is a solution only if m satisfies the quadratic equation

$$m(m-1) + am + b = 0$$

Or

$$m^2 + (a-1)m + b = 0$$

Roots of this equation are

$$m_1 = -\frac{a-1}{2} + \sqrt{\left(\frac{a-1}{2}\right)^2 - b} \qquad m_2 = -\frac{a-1}{2} - \sqrt{\left(\frac{a-1}{2}\right)^2 - b}$$

2.4. Euler-Cauchy equations

Three cases are possible

Case 1: $(a-1)^2 > 4b$, Two distinct real roots, $m_1 \neq m_2$

Partial solutions take the form $y_1(x) = x^{m_1}$ and $y_2(x) = x^{m_2}$. The general solution takes the form

$$y(x) = c_1 x^{m_1} + c_2 x^{m_2} \tag{2.4.2}$$

Case 2: $(a-1)^2 = 4b$, Double real root $m = m_1 = m_2 = -(a-1)/2$

One partial solution is $y_1(x) = x^m$. Another linearly independent particular solution can be found from Liouville theorem:

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1^2(x)} dx = x^m \int \frac{e^{-\int (a/x)dx}}{x^{2m}} dx = x^m \int x^{-a-2m}\, dx = x^m \ln x$$

Then the general solution takes the form

$$y(x) = c_1 x^m + c_2 \ln x\, x^m = (c_1 + c_2 \ln x) x^m \tag{2.4.3}$$

Case 3: $(a-1)^2 < 4b$, Two distinct complex roots

$$m_1 = \alpha + i\beta \qquad m_2 = \alpha - i\beta \qquad \alpha = -\frac{a-1}{2} \qquad \beta = \sqrt{b - \left(\frac{a-1}{2}\right)^2}$$

2.4. Euler-Cauchy equations

The general solution takes the form

$$y(x) = c_1 x^{m_1} + c_2 x^{m_2} = c_1 e^{\ln x^{\alpha+i\beta}} + c_2 e^{\ln x^{\alpha-i\beta}} = x^{\alpha}\left(c_1 e^{i\beta \ln x} \quad + c_2 e^{-i\beta \ln x} \quad\right)$$
$$= x^{\alpha}[c_1(\cos\beta\ln x + i\sin\beta\ln x) + c_2(\cos\beta\ln x - i\sin\beta\ln x)]$$

This equation gives *real-valued* solution only if c_1 and c_2 are *complex* numbers as well. Let's reduce the general solution to the form, containing *only real numbers*:

If we introduce $c_2 = (A + iB)/2$, $c_1 = (A - iB)/2$, then $c_1 + c_2 = A$, $i(c_1 - c_2) = -i^2 B = B$. Thus, the general solution takes the form

$$y(x) = x^{\alpha}[A\cos(\beta\ln x) + B\sin(\beta\ln x)] \qquad (2.4.4)$$

Example 1: Important examples of the Euler-Cauchy equation are related to the thermal transport in spherical symmetry. Spherically symmetric steady-state heat transfer equation

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0 \quad \Longrightarrow \quad r^2\frac{d^2T}{dr^2} + 2r\frac{dT}{dr} = 0 \qquad : \text{Euler} - \text{Cauchy equation}$$

See example 2 in Section 2.1.

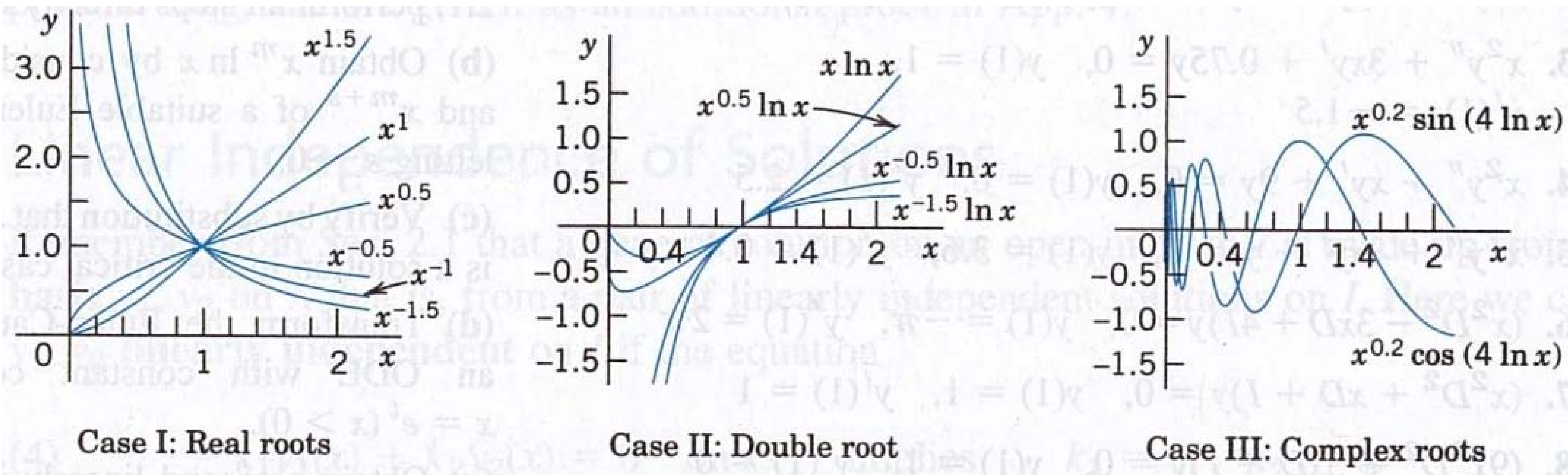
Example 2: Typical basis functions for the Euler-Cauchy equations.

1. $x^2y'' + 1.5xy' - 0.5y = 0$, Case 1, Two real roots, $y(x) = c_1\sqrt{x} + \frac{c_2}{x}$

2. $x^2y'' + axy' + 0.25(1-a)^2y = 0$, Case 2, Double real root, $y(x) = (c_1 + c_2 \ln x)x^{\frac{1-a}{2}}$

3. $x^2y'' + 0.6xy' + 16.04y = 0$, Case 3, Two complex roots,

$$y(x) = x^{0.2}[A\cos(4\ln x) + B\sin(4\ln x)]$$



2.5. Second-order linear nonhomogeneous ODEs. Method of undetermined coefficients

Our goal is to study the general properties of solutions of the linear nonhomogeneous equations of the 2nd order

$$y'' + p(x)y' + q(x)y = r(x) \qquad (2.5.1)$$

Assume that $y_p(x)$ is some particular solution of Eq. (2.5.1). Along with this solution, let's consider corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \qquad (2.5.2)$$

and let $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$ be the general solution of the homogeneous equation.

Theorem: **The structure of the general solution of a nonhomogeneous linear ODE**

If coefficients $p(x)$, $q(x)$, and $r(x)$ in Eq. (2.5.1) are continuous functions in some open interval (a, b) then the general solution of Eq. (2.5.1) has the form

$$y(x) = y_h(x) + y_p(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x) \qquad (2.5.3)$$

The general solution (2.5.3) includes all solutions, i.e. Eq. (2.5.1) has no singular solutions, and any solution can be represented in the form (2.5.3).

Proof:

Let's first check that the RHS of Eq. (2.5.3) is a solution of (2.5.1). Substituting Eq. (2.5.3) into Eq. (2.5.1) one can find that equation becomes identity.

2.5. Second-order linear nonhomogeneous ODEs. Method of undetermined coefficients

Second, assume that we have some solution $y^*(x)$ of the Cauchy problem for Eq. (2.5.1) with the initial conditions $y(x_0) = {y_0}^*$ and $y'(x_0) = {y_0}'^*$. Then $y^*(x) - y_p(x)$ is a solution of Eq. (2.5.2) (can be proved by substitution). From Theorem 4 about the structure of the general solution of a homogeneous equation, $y^*(x) - y_p(x) = \mathrm{c}_1 y_1(x) + \mathrm{c}_2 y_2(x)$. This solution satisfies the following initial conditions:

$$
\begin{aligned}
{y_0}^* - y_p(x_0) &= \mathrm{c}_1 y_1(x_0) + \mathrm{c}_2 y_2(x_0) \\
{y_0}'^* - {y_p}'(x_0) &= \mathrm{c}_1 {y_1}'(x_0) + \mathrm{c}_2 {y_2}'(x_0)
\end{aligned}
$$

In this linear system, c_1 and c_2 are unknown. The determinant of the matrix of coefficients is $W(y_1, y_2, x_0) \neq 0$ and, thus, this system has a unique solution. We proved that for any ${y_0}^*$ and ${y_0}'^*$ solution of the nonhomogeneous equation can be represented in the form of Eq. (2.5.3).

Note: The algorithm of solving nonhomogeneous linear ODEs is a consequence of this theorem:

1. Find the general solution of the corresponding homogeneous equation $y_h(x) = \mathrm{c}_1 y_1(x) + \mathrm{c}_2 y_2(x)$.
2. Find some (just one) particular solution $y_p(x)$ of non-homogeneous equation. Then the general solution is given by Eq. (2.5.3).

There are two general approaches for finding $y_p(x)$:

1. Method of undetermined coefficients.
2. Lagrange method of variation of parameters.

2.5. Second-order linear nonhomogeneous ODEs. Method of undetermined coefficients

Method of undetermined coefficients

- Applied to non-homogeneous equations with constant coefficients

$$y'' + ay' + by = r(x) \qquad (2.5.4)$$

- Applied if the RHS $r(x)$ has a special form that includes functions with "self-similar" derivatives like $e^x, x^n, \cos x$, etc.
- The particular solution can be found by a trial-and-error approach with gradually increasing complexity of guess functions. These functions contain undetermined coefficients, which should be determined from the condition that the guess function satisfies the equation.
- The method, as described in Kreyszig's textbook, Sect.2.7, pages 81-82, is summarized below.

(a) Basic Rule. *If $r(x)$ in (4) is one of the functions in the first column in Table 2.1, choose y_p in the same line and determine its undetermined coefficients by substituting y_p and its derivatives into (4).*

(b) Modification Rule. *If a term in your choice for y_p happens to be a solution of the homogeneous ODE corresponding to (4), multiply this term by x (or by x^2 if this solution corresponds to a double root of the characteristic equation of the homogeneous ODE).*

(c) Sum Rule. *If $r(x)$ is a sum of functions in the first column of Table 2.1, choose for y_p the sum of the functions in the corresponding lines of the second column.*

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$ $k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$ $ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

Here C, K_i, K, M are coefficients that are determined not by the initial conditions, but by the parameters of the equation itself.

Method of undetermined coefficients: A basic particular case

Let's see how this method works for the equation

$$y'' + ay' + by = r(x) \tag{2.5.4}$$

with the RHS in the form

$$r(x) = e^{\gamma x}\left(P_l(x)\cos\omega x + Q_m(x)\sin\omega x\right) \tag{2.5.5}$$

where $P_l(x)$ and $Q_m(x)$ are **polynomials** of degrees l and m, i.e.

$$P_l(x) = C_{P,0} + C_{P,1}x + \cdots + C_{P,l}x^l$$

Algorithm:

Step 1. Let's solve the characteristic equation for the corresponding homogeneous equation, $\lambda^2 + a\lambda + b = 0$, and find its roots λ_1 and λ_2.

Step 2. Let's introduce the complex number $z = \gamma + i\omega$ and an integer k such that

$$k = \begin{cases} 0 & \text{if } z \text{ is not a root of the charcteristic equation} \\ 1 & \text{if } z \text{ coincides with one distinct root of the char. eq.} \\ 2 & \text{if } z \text{ coincides with the double root of the char. eq.} \end{cases}$$

Step 3. Let's define $n = \max(l, m)$ and look for a particular solution in the form

$$\boxed{y_p(x) = x^k e^{\gamma x}\left(\widetilde{P_n}(x)\cos\omega x + \widetilde{Q_n}(x)\sin\omega x\right)} \tag{2.5.6}$$

Step 4. Every polynomial $\widetilde{P_n}(x)$ and $\widetilde{Q_n}(x)$ of degree n contains $n+1$ undetermined coefficients that should be defined by substituting Eq. (2.5.6) into Eq. (2.5.4).

2.5. Second-order linear nonhomogeneous ODEs. Method of undetermined coefficients

Example 1: $y'' - 3y' + 2y = xe^{-x}$.

Step 1. $\lambda^2 - 3\lambda + 2 = 0$, $\lambda_1 = 2$, $\lambda_2 = 1$.

Step 2. $\gamma = -1$, $\omega = 0$, $z = -1$, $k = 0$.

Step 3. $n = \max(1,0) = 1$, so that we are looking for a particular solution in the form

$$y_p(x) = e^{-x}(Ax + B)$$

where A and B are two undetermined coefficients in the polynomial of degree $n = 1$.

Step 4. These coefficients are not arbitrary, but can be found from the condition that $y_p(x)$ is a solution of the nonhomogeneous equation:

$$y_p' = -e^{-x}(Ax + B) + e^{-x}A$$
$$y_p'' = e^{-x}(Ax + B) - 2e^{-x}A$$

After inserting $y_p(x)$ into equation, e^{-x} can be dropped in every term, and the rest gives

$$(Ax + B) - 2A - 3(-Ax - B + A) + 2(Ax + B) = x$$

In order to obtain identity, the groups of terms at every degree of x should be equal to zero:

If the choice of the guess function for $y_p(x)$ is correct, then the number of linear equations is equal to the number of undetermined coefficients A, B, etc.

At x^0: $A + 3A + 2A = 1$, i.e. $A = 1/6$.

At x^1: $B - 2A + 3B - 3A + 2B = 0$, i.e. $6B = 5A, B = 5/36$.

Solution: $y_p(x) = e^{-x}(x + 5/6)/6$, $y(x) = c_1 e^x + c_2 e^{2x} + e^{-x}(x + 5/6)/6$.

Example 2: $y'' + 4y = 2\cos 2x$.

Step 1. $\lambda^2 + 4 = 0$, $\lambda_1 = 2i$, $\lambda_2 = -2i$.

Step 2. $\gamma = 0$, $\omega = 2$, $z = i2 = \lambda_1$, $k = 1$.

Step 3. $n = \max(0{,}0) = 0$, so that we are looking for a particular solution in the form

$$y_p(x) = x(A\cos 2x + B\sin 2x)$$

where A and B are two undetermined coefficients in the two polynomials of degree $n = 0$.

Step 4. These coefficients are not arbitrary, but should be found from the condition that $y_p(x)$ is a solution of the nonhomogeneous equation:

$$y_p' = A\cos 2x + B\sin 2x + 2x(-A\sin\ 2x + B\cos 2x)$$

$$y_p'' = 4(-A\sin\ 2x + B\cos 2x) - 4x(\cos 2x + B\sin 2x) = 4(-A\sin\ 2x + B\cos 2x) - 4y_p$$

After substituting $y_p(x)$ into equation, one can obtain

$$4(-A\sin\ 2x + B\cos 2x) = 2\cos 2x$$

In order to have identity, the groups of terms at $\sin 2x$ and $\cos 2x$ should be equal to zero:

At $\sin 2x$: $-4A = 0$, i.e. $A = 0$.

At $\cos 2x$: $4B = 2$, i.e. $B = 1/2$.

Solution: $y_p(x) = (x/2)\sin 2x$, $y(x) = C\cos(2x - \varphi) + (x/2)\sin 2x$.

2.6. Second-order linear nonhomogeneous ODEs. Method of variation of parameters

Lagrange method of variation of parameters

Let's consider the linear nonhomogeneous ODE of the 2nd order is the standard form:

$$y'' + p(x)y' + q(x)y = r(x) \qquad (2.6.1)$$

and assume that $y_0(x) = c_1 y_1(x) + c_2 y_2(x)$ is the general solution of the corresponding homogeneous equation.

Assumption of the Lagrange method: Let's try to find the *general solution* of the homogeneous equation in the same form as for the homogeneous ODE, but assuming now that c_1 and c_2 are functions of x, i.e.

$$y(x) = c_1(x)y_1(x) + c_2(x)y_2(x) \qquad (2.6.2)$$

Now we have two unknown functions, $c_1(x)$ and $c_2(x)$, but after inserting (2.6.2) into (2.6.1) we can obtained only one condition to which both functions should satisfy. In order to find two functions, we need two conditions. Let' choose the second condition in the form that brings derivative of (2.6.2) to the same form as in the case of the general solution of the homogeneous equation. In other words, let's look for $c_1(x)$ and $c_2(x)$ that additionally satisfy the condition

$$c_1' y_1 + c_2' y_2 = 0.$$

Then

$$y' = \cancel{c_1' y_1} + c_1\ y_1' + \cancel{c_2' y_2} + c_2\ y_2' = c_1\ y_1' + c_2\ y_2'$$

$$y'' = c_1' y'_1 + c_1\ y_1'' + c'_2\ y'_2 + c_2\ y_2''$$

Inserting these derivatives into Eq. (2.6.1), one can find that

2.6. Second-order linear nonhomogeneous ODEs. Method of variation of parameters

Lagrange method of variation of parameters

$$c_1' y'_1 + \cancel{c_1 y_1''} + c'_2\, y'_2 + \cancel{c_2\, y_2''} + \cancel{p(c_1 y'_1 + c_2 y'_2)} + \cancel{q(c_1 y_1 + c_2 y_2)} = r$$

Many terms in this equation cancel each other since y_1 and y_2 are solutions of the homogeneous equation.

Now we see that $c_1(x)$ and $c_2(x)$ should be found as solutions of two differential equations

$$c_1' y_1 + c_2' y_2 = 0.$$

$$c_1' y'_1 + c'_2\, y'_2 = r$$

which can be re-written in the vector form as

$$\mathbf{Mx} = \mathbf{b}, \quad \mathbf{M} = \begin{pmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} c'_1(x) \\ c'_2(x) \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ r(x) \end{pmatrix}$$

The determinant of the matrix of coefficients is the Wronskian of two linearly independent solutions, so that $W(y_1, y_2, x) \neq 0$, inverse matrix $\mathbf{M}^{-1}$ exists, and the linear system has a unique solution (for any x from an interval, where y_1 and y_2 are linearly independent solutions):

$$\mathbf{M}^{-1} = \frac{1}{W}\begin{pmatrix} y'_2 & -y_2 \\ -y'_1 & y_1 \end{pmatrix} \quad \Longrightarrow \quad \mathbf{x} = \mathbf{M}^{-1}\mathbf{b} = \frac{1}{W}\begin{pmatrix} -y_2 r \\ y_1 r \end{pmatrix}.$$

We see that equations for $c_1(x)$ and $c_2(x)$ are separated:

$$c_1' = -\frac{y_2 r}{W}, \qquad c_2' = \frac{y_1 r}{W}$$

RHSs in these equations are known functions of x, so that these equations can be easily solved:

2.6. Second-order linear nonhomogeneous ODEs. Method of variation of parameters

Lagrange method of variation of parameters

$$\boxed{c_1\ (x) = -\int \frac{y_2(x)r(x)}{W(x)}dx + \bar{c}_1, \qquad c_2\ (x) = \int \frac{y_1(x)r(x)}{W(x)}dx + \bar{c}_2} \qquad (2.6.3)$$

Substituting Eq. (2.6.3) into (2.6.2), one can write the solution of the non-homogeneous equation in the form

$$y(x) = \underbrace{\bar{c}_1 y_1(x) + \bar{c}_2 y_2(x)}_{y_h(x)} + \underbrace{y_2(x)\int \frac{y_1(x)r(x)}{W(x)}dx - y_1(x)\int \frac{y_2(x)r(x)}{W(x)}dx}_{y_p(x)} \qquad (2.6.4)$$

i.e. the Lagrange method allows us to find the general solution of the non-homogeneous equation or to represent the solution in the form given by Eq. (2.5.3) and find a particular solution of the non-homogeneous equation in the form

$$y_p(x) = y_2(x)\int \frac{y_1(x)r(x)}{W(x)}dx - y_1(x)\int \frac{y_2(x)r(x)}{W(x)}dx \qquad (2.6.5)$$

2.6. Second-order linear nonhomogeneous ODEs. Method of variation of parameters

Example: $y'' - 2y' + y = e^x/(x^2+1)$: The method of undetermined coefficients can not be used.

Step 1. We need to find two linearly independent solutions of the corresponding homogeneous equation: $\lambda^2 - 2\lambda + 1 = 0$, $\lambda = \lambda_1 = \lambda_2 = 1$, $y_1(x) = e^x$, $y_2(x) = xe^x$.

Step 2. We need to calculate the Wronskian:

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}$$

Step 3. We can directly apply Eqs. (2.6.3):

$$c_1\ (x) = -\int \frac{y_2(x)r(x)}{W(x)} dx + \bar{c}_1 = -\int \frac{xe^x}{e^{2x}} \frac{e^x}{x^2+1} dx + \bar{c}_1 = -\frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} + \bar{c}_1$$

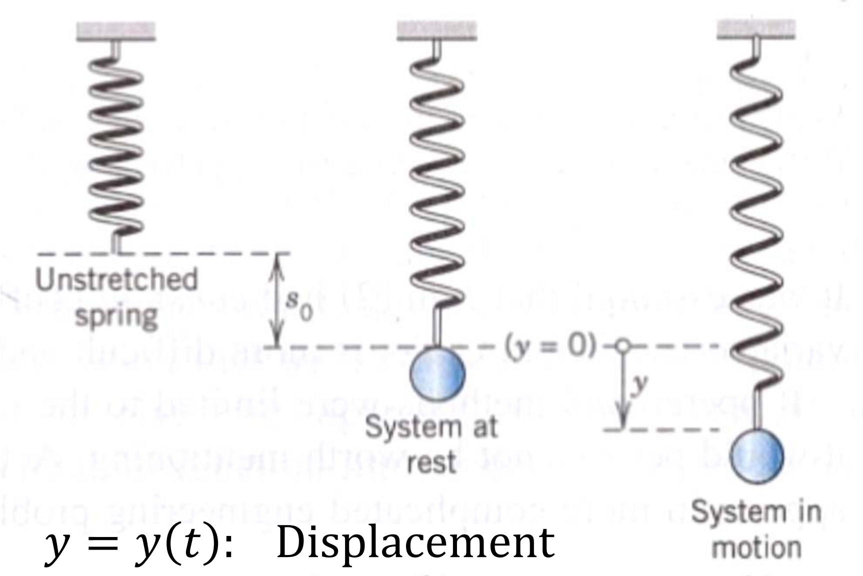
$$= -\frac{\ln(x^2+1)}{2} + \bar{c}_1$$

$$c_2\ (x) = \int \frac{y_1(x)r(x)}{W(x)} dx + \bar{c}_2 = \int \frac{e^x}{e^{2x}} \frac{e^x}{x^2+1} dx + \bar{c}_2 = \int \frac{dx}{x^2+1} = \arctan x + \bar{c}_2$$

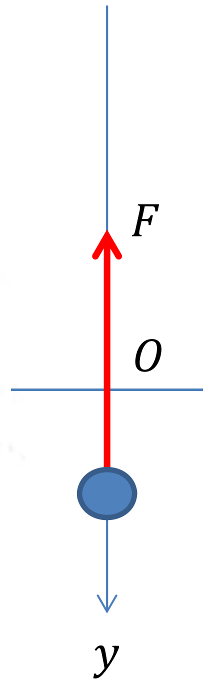
Step 4: The general solution of the nonhomogeneous equation is given by Eq. (2.6.4), i.e.

$$y(x) = \left(-\frac{\ln(x^2+1)}{2} + \bar{c}_1\right) e^x + (\arctan x + \bar{c}_2) x e^x$$

Mechanical mass-spring system



$y = y(t)$: Displacement



Newton's second law of motion:

$$my'' = \sum F_i = F_1 + F_2 + F_3$$

1. **Elastic restoring force** (Hook's law):

$$F_1 = -ky$$

k is the **spring constant** (spring stiffness)

2. **Damping (friction) force**:

$$F_2 = -cy'$$

3. **Input (driving) external force**

$$F_3 = r(t)$$

$$\boxed{my'' + cy' + ky = r(t)}$$

Undamped oscillation : $c = 0$
Damped oscillation: $c \neq 0$

Free oscillation: $r(t) = 0$
Forced oscillation: $r(t) \neq 0$

2.7. Free oscillations in mechanical systems

I. Undamped free oscillation

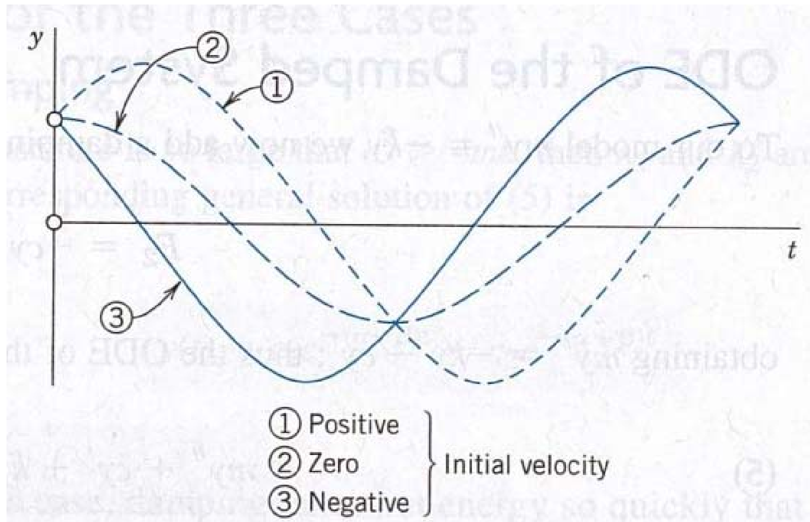
$$my'' + ky = 0$$

$$y'' + {\omega_0}^2 y = 0 \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

$$\lambda_{1,2} = \pm i\omega_0$$

$$y(t) = A\cos\omega_0 t + B\sin\omega_0 t = C\cos(\omega_0 t - \delta)$$

This type of motion is called **harmonic oscillation:**



C, Magnitude (m)

ω_0, **Natural angular frequency**

$f_0 = \omega_0/(2\pi)$, **Natural frequency** (Hz)

$\tau_0 = 1/f_0$, Period of oscillation (s)

δ, Phase

The initial conditions, $y(0) = y_0$ and $y'_0(0) = v_0$, allow one to determine unique C and δ, while frequency of oscillation is determined by the properties of the system and does not depend on the initial conditions.

II. Damped free oscillations

$$my'' + cy' + ky = 0$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$y'' + \frac{c}{m}y' + \omega_0^{\,2} y = 0$$

$$\alpha = \frac{c}{2m}$$

$$\lambda^2 + \frac{c}{m}\lambda + \omega_0^{\,2} = 0, \qquad \lambda_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^{\,2}}$$

Case II.1: $c^2 > 4m^2\omega_0^{\,2}$, Overdamping:

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

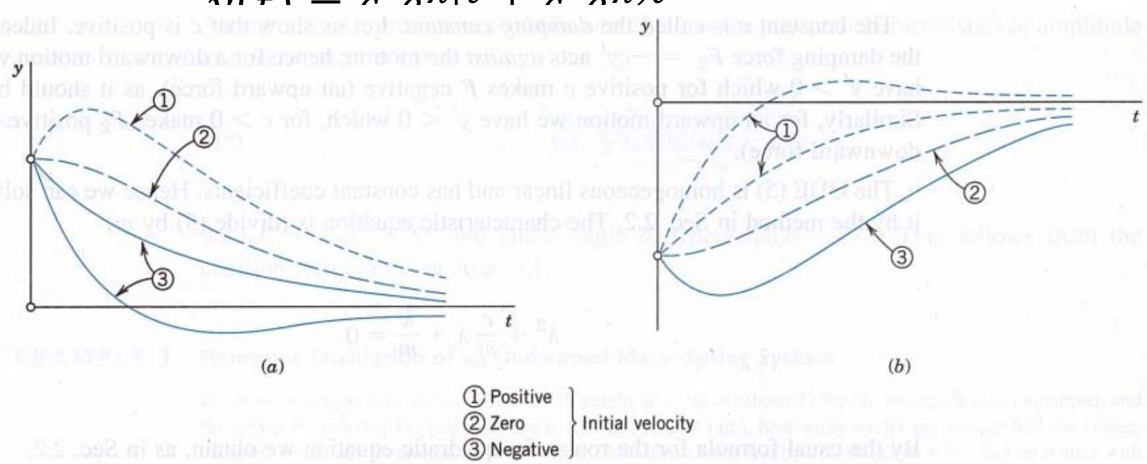


Fig. 37. Typical motions (7) in the overdamped case
(*a*) Positive initial displacement
(*b*) Negative initial displacement

Case II.2: $c^2 = 4m^2\omega_0^{\,2}$, Critical damping:

$$y(t) = (c_1 + c_2 t)e^{-\alpha t}$$

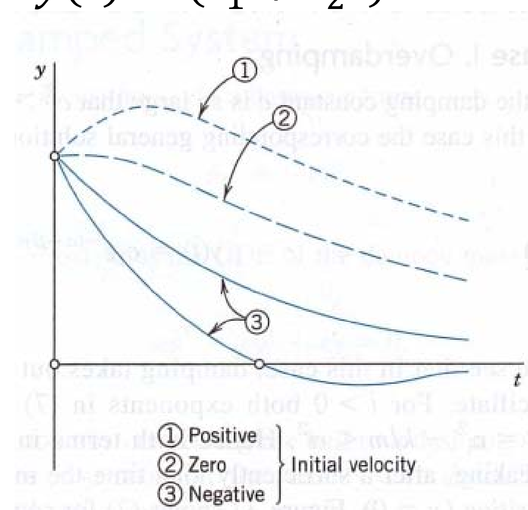


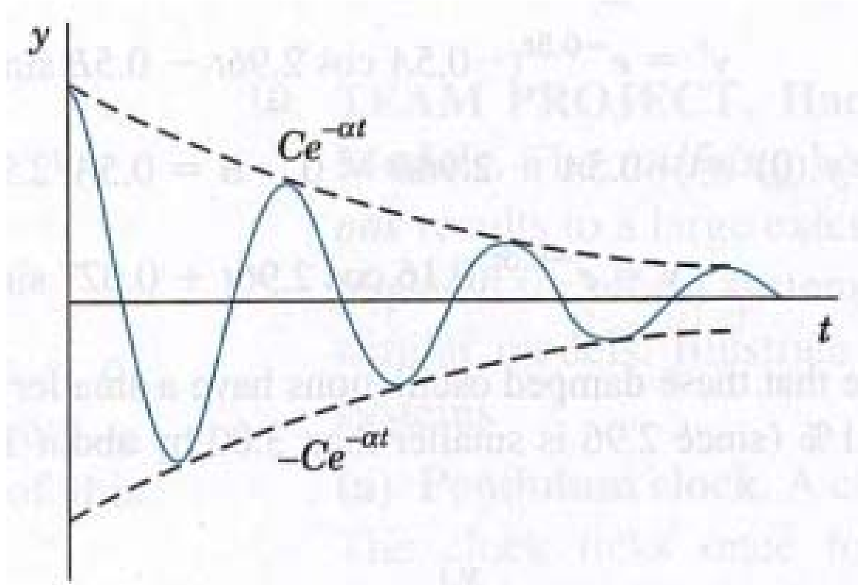
Fig. 38. Critical damping [see (8)]

2.7. Free oscillations in mechanical systems

Case II.3: $c^2 < 4m^2\omega_0{}^2$, Underdamping

$$\omega_* = \sqrt{\omega_0{}^2 - \alpha^2} = \omega_0\sqrt{1 - \left(\frac{\alpha}{\omega_0}\right)^2} < \omega_0$$

$$y(t) = Ce^{-\alpha t}\cos\left(\omega_*\, t - \delta\right)$$

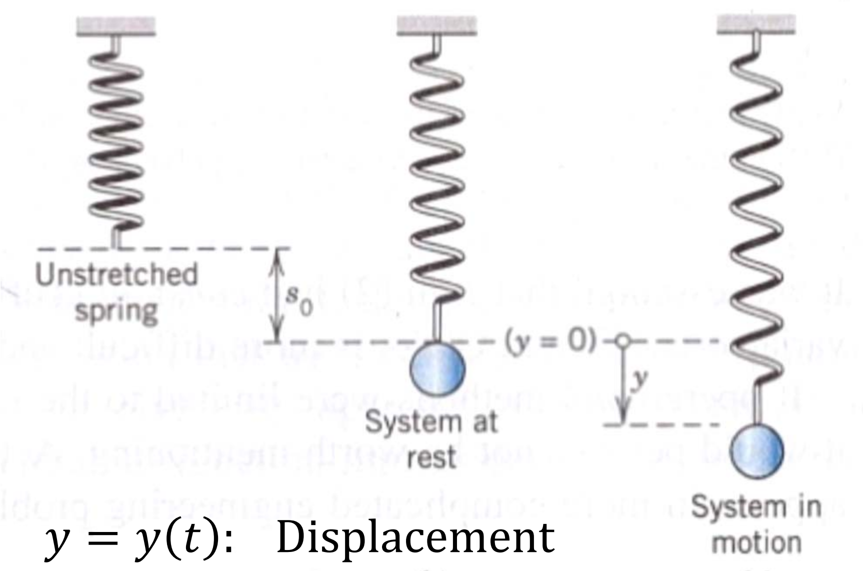


Note 1: Frequency of damped oscillation does not coincide with the natural frequency of the system.

Note 2: In all three cases, since $\alpha > 0$, $y(t) \to 0$ when $t \to \infty$. In the damped system, after a sufficiently long time, the free oscillating mass will be at rest at its static equilibrium position.

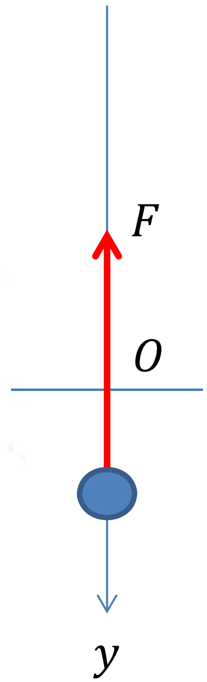
2.8. Forced oscillations and resonance in mechanical systems

Mechanical mass-spring system



$y = y(t)$: Displacement

$$\boxed{my'' + cy' + ky = r(t)}$$



Newton's second law of motion:

$$my'' = \sum F_i = F_1 + F_2 + F_3$$

1. **Elastic restoring force** (Hook's law):

$$F_1 = -ky$$

k is the **spring constant** (spring stiffness)

2. **Damping (friction) force**:

$$F_2 = -cy'$$

3. **Input (driving) external force**

$$F_3 = r(t)$$

III. Forced oscillation

We consider only a **harmonic driving force**, $r(t) = F_0 \cos \omega\, t$, ω is the **input angular frequency**

$$my'' + cy' + ky = F_0 \cos \omega\, t \qquad (2.8.1)$$

$$y(t) = y_h(t) + y_p(t)$$

Let's use the method of undetermined coefficients in order to find a particular solution $y_p(t)$ of this nonhomogeneous linear ODE.

2.8. Forced oscillations and resonance in mechanical systems

Let's consider the case when $\boldsymbol{\omega_0 \neq \omega}$

According to our *basic particular case of the method of undetermined coefficients*,

$$y_p(t) = a\cos\omega t + b\sin\omega t \tag{2.8.2}$$

where a and b are two coefficient that should be defined from the ODE. Since

$$y'_p = \omega(-a\sin\omega t + b\cos\omega t), \qquad y''_p = -\omega^2(a\cos\omega t + b\sin\omega t) = -\omega^2\, y_p$$

Substituting y_p into the ODE, one can obtain:

$$-\omega^2 y_p + \frac{c}{m}\omega(-a\sin\omega t + b\cos\omega t) + \omega_0^2 y_p = \frac{F_0}{m}\cos\omega\, t$$

In order to obtain identity, coefficients at $\cos\omega t$ and $\sin\omega t$ should be equal to zero:

At $\cos\omega t$: $\quad (\omega_0^2 - \omega^2)a + \frac{c}{m}b\omega = \frac{F_0}{m}$

:Two equations with respect to a and b

At $\sin\omega t$: $\quad (\omega_0^2 - \omega^2)b - \frac{c}{m}a\omega = 0$

If $\omega \neq \omega_0$, then the solution of these equations is

$$a = F_0\frac{m(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 m^2 + c^2\omega^2}, \qquad b = F_0\frac{\omega c}{(\omega_0^2 - \omega^2)^2 m^2 + c^2\omega^2} \tag{2.8.3}$$

2.8. Forced oscillations and resonance in mechanical systems

Case III.1: $c = 0$, Undamped forced oscillations

$$a = \frac{F_0}{(\omega_0^2 - \omega^2)m}, \qquad b = 0,$$

$$y(t) = C\cos(\omega_0 t - \delta) + \frac{F_0}{(\omega_0^2 - \omega^2)m}\cos\omega t, \tag{2.8.4}$$

- Undamped forced oscillation is a superposition of two harmonic oscillations *with different frequencies*.
- In the first term, the magnitude C is determined by the initial conditions.
- In the second term, the magnitude does not depend on the initial conditions and can be made arbitrarily large if $\omega \to \omega_0$.

Excitation of large-magnitude oscillations by matching input and natural frequencies is called the **resonance**. In the case of resonance, $\omega = \omega_0$, and Eqs. (2.8.2)-(2.8.4) *are no longer valid*. According to our basic particular case, if $\omega = \omega_0$, then $k = 1$ and we should look for a particular solution of the nonhomogeneous ODE in the form

$$y_p(t) = t(a\cos\omega_0 t + b\sin\omega_0 t)$$

and then

$$y'_p = a\cos\omega_0 t + b\sin\omega_0 t + t\omega_0(-a\sin\omega_0 t + b\cos\omega_0 t)$$
$$y''_p = 2\omega_0(-a\sin\omega_0 t + b\cos\omega_0 t) - {\omega_0}^2 y_p$$

Coefficients a and b should turn the equation

$$2\omega_0(-a\sin\omega_0 t + b\cos\omega_0 t) - \cancel{{\omega_0}^2 y_p} + \cancel{{\omega_0}^2 y_p} = (F_0/m)\cos\omega_0 t$$

2.8. Forced oscillations and resonance in mechanical systems

into identity. It is possible only if $a = 0$ and $b = F_0/(2\omega_0 m)$. Then the particular solution of the nonhomogeneous ODE at resonance has the form

$$y_p(t) = \frac{F_0}{2\omega_0 m} t \sin \omega_0 t$$

This result shows that *the magnitude of forced oscillation at resonance linearly increases with time.*

If $|\omega_0 - \omega| \ll \omega$, but $\omega_0 \neq \omega$, then this type of motion is called **beats**. The general solution takes the form

$$y(t) = C \cos(\omega_0 t - \delta) + \frac{F_0}{(\omega_0^2 - \omega^2)m} \cos \omega t$$

For the sake of simplicity, let's consider the only case when initial conditions correspond to $C = -F_0/((\omega_0^2 - \omega^2)m)$ and $\delta = 0$. Then

$$y(t) = -C(\cos \omega t - \cos \omega_0 t)$$

Now let's show that this solution corresponds to a "product" of two oscillations with very different frequencies. For this purpose we can use the following property of trigonometric functions:

$$\cos(x - y) - \cos(x + y) = \cos x \cos y + \sin x \sin y - \cos x \cos y + \sin x \sin y = 2 \sin x \ \sin y$$

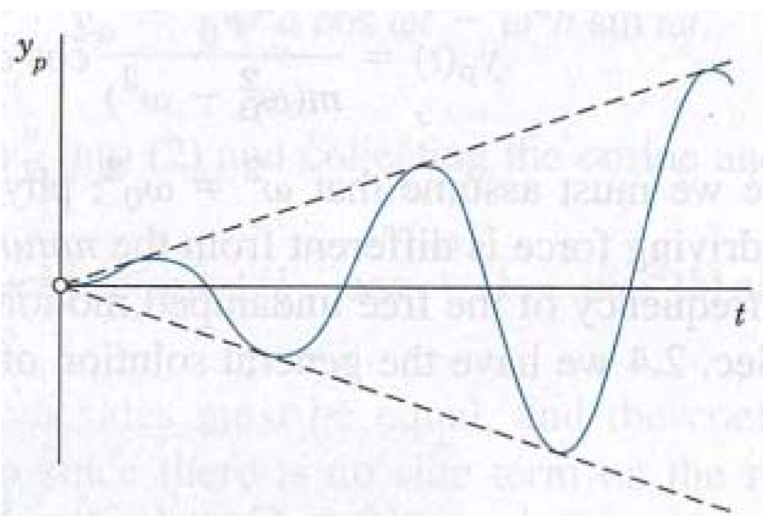
2.8. Forced oscillations and resonance in mechanical systems

Now, if we introduce $\omega_a = (\omega_0 + \omega)/2$ and $\omega_b = (\omega_0 - \omega)/2$ (and, thus, $\omega_b - \omega_a = \omega$, $\omega_b + \omega_a = \omega_0$), we can re-write the solution in the form

$$y(t) = \frac{2F_0}{(\omega_0^2 - \omega^2)m} \sin \omega_a t \sin \omega_b t = \frac{2F_0}{(\omega_0^2 - \omega^2)m} \sin \frac{(\omega_0 + \omega)t}{2} \sin \frac{(\omega_0 - \omega)t}{2}$$

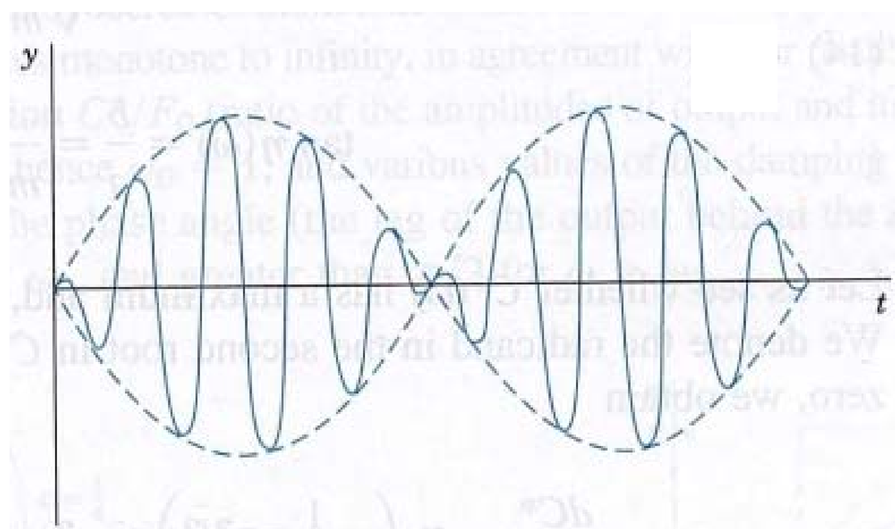
In the case of beats, $\omega_a / \omega_b = (\omega_0 + \omega)/(\omega_0 - \omega) \sim 2\omega/(\omega_0 - \omega) \gg \omega$.

Resonance



$\omega_0 = \omega$

Beats



$|\omega_0 - \omega| \ll \omega$, but $\omega_0 \neq \omega$

2.8. Forced oscillations and resonance in mechanical systems

- Resonance and beats are general phenomena specific for numerous oscillating systems.
- Sometimes resonance is useful, e.g. the resonant amplification is used for registering signals of small magnitudes (radio).
- In engineering applications, resonance and beats are often dangerous phenomena that result in reduced durability and/or catastrophic failures of engineering designs.

Tacoma Narrows bridge collapse (http://en.wikipedia.org/wiki/Tacoma_Narrows_Bridges)

Opening day, July 1, 1940

Collapse, November 7, 1940

- In order to avoid resonance and beats, the oscillating systems should be designed providing large difference between natural frequency(s) of the system and input frequencies of various external forces.

Case III.2: $c \neq 0$, Damped forced oscillation

$$y(t) = y_h(t) + y_p(t)$$

In the case of damped oscillation, $y_h(t) \to 0$ when $t \to \infty$, so that

$$y(t) \to y_p(t)$$

i.e., after a sufficiently long time, the solution approaches a "steady-state" solution given only by the particular solution of the nonhomogeneous equation.

In the steady-state, the magnitude of oscillation remains constant, but it can be very large if ω is close to natural frequency ω_0. The case when ω provides maximum magnitude at given ω_0 is called the **practical resonance**.

The maximum magnitude of oscillation at practical resonance occurs *if the input frequency is close but does not coincide with the natural frequency.* Let's find the input frequency ω_{max} which corresponds to the maximum magnitude of oscillation. Let's re-write Eq.(2.8.2)in the form

$$y_p(t) = C_* \cos(\omega t - \eta)$$

where the magnitude C_* and **phase lag** η are equal to (here we use Eq. (2.8.3))

$$C_* = \sqrt{a^2 + b^2} = \frac{F_0}{\sqrt{(\omega_0^2 - \omega^2)^2 m^2 + c^2 \omega^2}}, \qquad \tan\eta = \frac{b}{a}.$$

Now let's look for a maximum of $C_*(\omega)$. It is achieved when $dC_*/d\omega = 0$.

2.8. Forced oscillations and resonance in mechanical systems

$$\frac{dC_*}{d\omega} = -\frac{F_0}{2}\frac{-4\omega(\omega_0^2-\omega^2)\quad m^2+2c^2\omega}{[(\omega_0^2-\omega^2)^2m^2+c^2\omega^2]^{\frac{3}{2}}}, \qquad (2.8.5)$$

We see that $dC_*/d\omega = 0$ at $\omega = \omega_{max}$, where $\omega_{max} \neq 0$ should turn the numerator in the RHS of Eq. (2.8.5) to zero:

$$-4(\omega_0^2 - \omega_{max}{}^2)\quad m^2 + 2c^2 = 0$$

$$\omega_{max}{}^2 = \omega_0{}^2 - \frac{c^2}{2m^2} < \omega_0{}^2$$

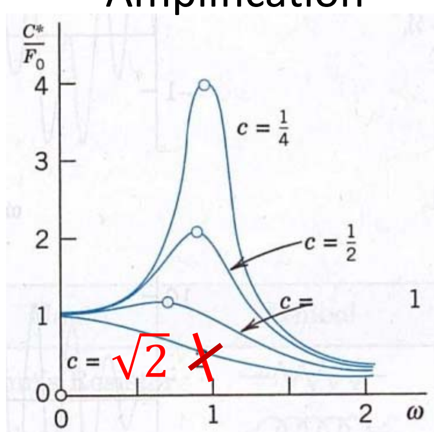
The maximum magnitude at practical resonance is equal to

$$C_{max} = C_*(\omega_{max}) = \frac{2mF_0}{c\sqrt{4\omega_0{}^2m^2 - c^2}}$$

Value C_*/F_0 is called the **amplification** (C_{max}/F_0 is the amplification at practical resonance).

$$m = 1$$
$$k = 1$$
$$\omega_0 = \sqrt{\frac{k}{m}} = 1$$

Amplification



Phase lag