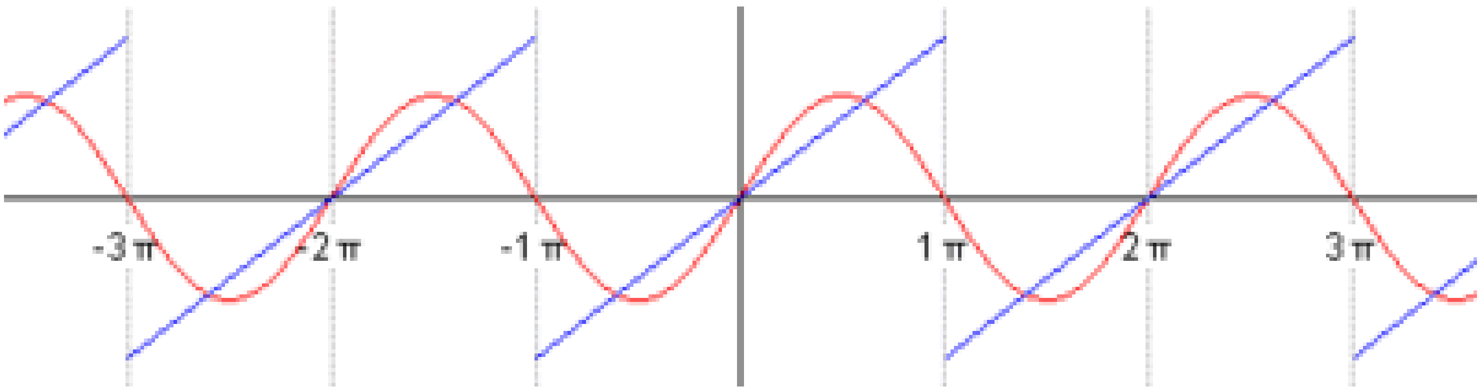
Chapter 5
Fourier Analysis

Reading:

Kreyszig, *Advanced Engineering Mathematics, 10th Ed.*, 2011
Selection from chapter 11

Prerequisites:

Kreyszig, *Advanced Engineering Mathematics, 10th Ed.*, 2011
Complex numbers: Sections 13.1, 13.2 and 13.5



Contents

Topic for self-studying: 5.16. Discrete Fourier transform (DFT). Fast Fourier transform (FFT)

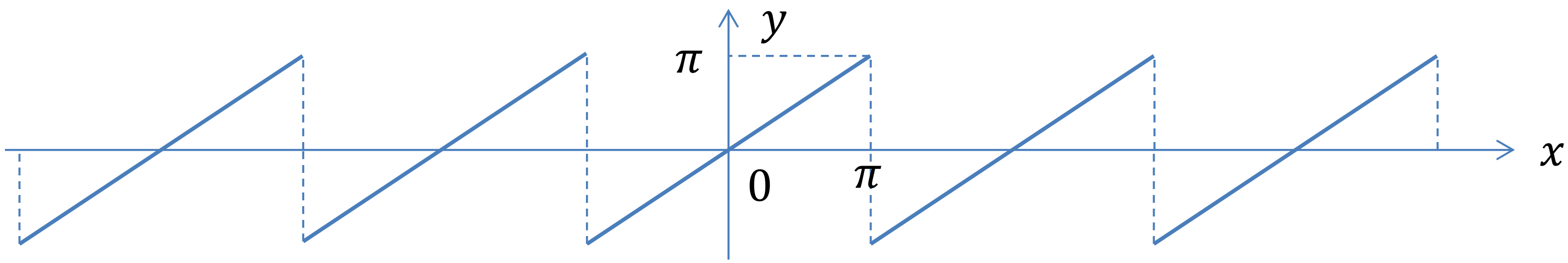
5.1. Fourier analysis. Motivation: Analysis of complex periodic and non-smooth functions

Analysis of complex functions is often based on their representation in the form a series - infinite sum of simple functions.

Example: Taylor expansion - Representation of a function $f(x)$ in the form of the power series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 +$$

Let's consider a periodic and non-smooth function



What if we will try to use the Taylor expansion?

In the point $a = 0$ we obtain the Taylor series in the form

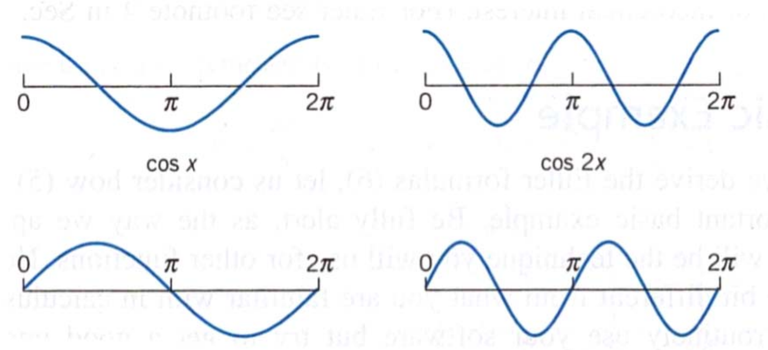
$$f(x) = x$$

This is great (accurate results for our $f(x)$) inside a period, but becomes meaningless outside the period since the function is discontinuous. The Taylor expansion can be applied only inside intervals where the function is continuous and has all continuous derivatives.

The major motivation of the Fourier analysis is to develop an approach for series representation of (almost arbitrary) discontinuous periodic and non-periodic functions.

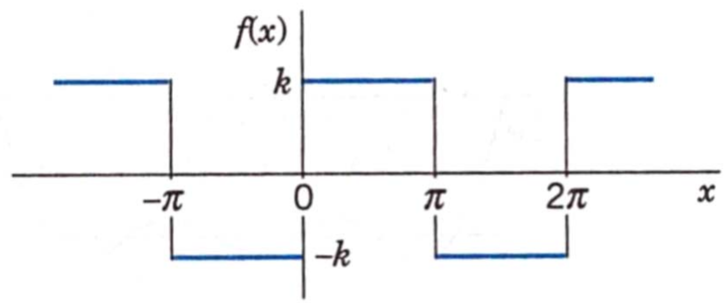
5.2. Periodic functions. Basic trigonometric function. Trigonometric sum and series

Many phenomena in science and engineering are periodic and described in terms of periodic functions.



Examples:

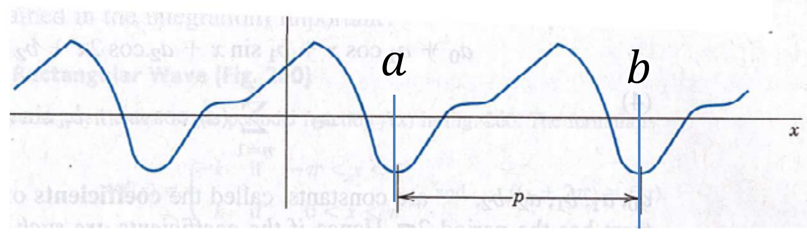
1. Mechanical oscillations (mass-spring systems, pendulums, strings, membranes).
2. Oscillations in electrical circuits.
3. Periodic motion of planets.
4. Wave motion (acoustic waves, electromagnetic waves, radio, etc.).
5. Oscillations of individual atoms in crystalline solids.



Periodic function $f(x)$ is a function which satisfies the following condition

$$\text{for all } x: \quad f(x+P) = f(x)$$

where parameter P is called the **period**.



Notes:

1. Obviously, if P is a period, then nP is also the period for function $f(x)$. By default, we will use the term period in order to denote the minimum period of function $f(x)$. The minimum period is also called the **fundamental period**.
2. It is sufficient to study any periodic function only at any interval $x \in [a, a+P]$.
3. Any function given at finite interval $x \in [a, b]$ can be **periodically extended** for any x with the period $P = b - a$.

5.2. Periodic functions. Basic trigonometric function. Trigonometric sum and series

Let's consider the functions for fixed parameter ω

$$F_n(x) = a_n \cos(n\omega x) + b_n \sin(n\omega x)$$

$F_n(x)$ is the **basic** trigonometric function for $n = 0,1,\ldots$ and a_n and b_n are constant **coefficients**.

Properties of $F_n(x)$:

1. Common period for all $F_n(x)$ is $P = 2\pi/\omega$, ω is the **fundamental** angular frequency.
   Proof: $F_n(x+P) = a_n \cos(n\omega x + 2\pi n) + b_n \sin(n\omega x + 2\pi n) = F_n(x)$
2. Let's introduce the **magnitude** $A_n = \sqrt{a_n^2 + b_n^2}$ and **phase** $\varphi_n = \arctan(b_n/a_n)$. Then

$$F_n(x) = A_n \cos(n\omega x - \varphi_n) = A_n \sin(n\omega x + \varphi_n)$$

   Proof: $A_n \cos(n\omega x - \varphi_n) = A_n(\cos\varphi_n \cos(n\omega x) + \sin\varphi_n \sin(n\omega x)) = F_n(x)$.
3. If $P = 2L$, L is the **half-period**, then $\omega = 2\pi/P = \pi/L$ and

$$F_n(x) = a_n \cos\frac{\pi n x}{L} + b_n \sin\frac{\pi n x}{L}$$

N-terms **trigonometric sum** is

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} F_n(x) = \frac{a_0}{2} + \sum_{n=1}^{N} [a_n \cos(n\omega x) + b_n \sin(n\omega x)]$$

Trigonometric series is

P is the fundamental period for $S_N(x)$ and $S(x)$

$$S(x) = \lim_{N\to\infty} S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} F_n(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega x) + b_n \sin(n\omega x)]$$

Two basic applications of the trigonometric sum and series

1. In many mathematical problems, trigonometric sum or series represents an accurate solution of the problem. In this case, ω and coefficients a_n and b_n are defined from equation to be solved and initial/boundary conditions.

Example: Sturm-Liouville problem (will be considered later)

2. Any periodic function with period P can be represented in the form of trigonometric series.

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} F_n(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega x) + b_n \sin(n\omega x)]$$

This is similar to the representation of a function in the form of a **power series**

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 +$$

where coefficients should coincide with the coefficients given by the **Taylor expansion**

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 +$$

In order to introduce the representation of a function in the form of a trigonometric series, we need to know how to define coefficients a_n and b_n (ω is not arbitrary, it is given by the period:

$$\boxed{\omega = 2\pi/P}$$

5.3. Orthogonal system of functions. Trigonometric system of functions

Let's consider a system of functions $f_0(x), f_1(x), f_2(x), \dots \ : f_m(x), \quad m = 0,1,2,\dots$ given at interval $x \in [a,b]$. The system of functions is called **orthogonal** in $[a,b]$ with respect to the **weight** $r(x) > 0$ if

$$(f_n, f_m) = \int_a^b r(x) f_n(x) f_m(x) dx = \begin{cases} 0, & n \neq m \\ \|f_m\|^2, & n = m \end{cases}$$

$$\|f_m\|^2 = \int_a^b r(x) f_m^2(x) dx > 0$$

The **trigonometric system** of functions at fixed ω is the system

$$1, \cos \omega x\,, \sin \omega x\,, \cos 2\omega x\,, \sin 2\omega x\,, \dots, \cos n\omega x\,, \sin n\omega x\,, \dots.$$

Theorem:
For a given ω, the trigonometric system of functions is orthogonal at any interval $[a, a+P]$, where $P = 2\pi/\omega$ with respect to weight $r(x) = 1$ and, moreover,

$$\|f_0\|^2 = P = 2L, \qquad \|f_m\|^2 = \frac{P}{2} = L \ \text{ at } \ m > 0. \tag{5.3.1}$$

Proof:

$$\|f_0\|^2 = \int_a^{a+P} 1^2 dx = P$$

5.3. Orthogonal system of functions. Trigonometric system of functions

In order to calculate other integrals, we need to use a number of trigonometric equations that all follow from

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$$
$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) \quad (5.3.2)$$

Let's first consider the case when $n=m$. Eq. (5.3.2) results in

$$\cos^2\alpha=\frac{1+\cos 2\alpha}{2}, \qquad \sin^2\alpha=\frac{1-\cos 2\alpha}{2}, \qquad \sin\alpha\cos\alpha=\frac{\sin 2\alpha}{2}$$

Then

$$\int_a^{a+P}\cos^2(n\omega x)dx=\int_a^{a+P}\frac{1+\cos(2n\omega x)}{2}dx=\frac{P}{2}$$

$$\int_a^{a+P}\sin^2(n\omega x)dx=\int_a^{a+P}\frac{1-\cos(2n\omega x)}{2}dx=\frac{P}{2}$$

$$\int_a^{a+P}\sin(n\omega x)\cos(n\omega x)\,dx=\frac{1}{2}\int_a^{a+P}\sin(2n\omega x)\,dx=0$$

5.3. Orthogonal system of functions. Trigonometric system of functions

Now let's consider $\int_a^{a+P} \cos(n\omega x)\cos(m\omega x)\,dx$ at $n \neq m$. Eq. (5.3.2) results in

$$\cos((n+m)\omega x) = \cos(n\omega x)\cos(m\omega x) - \sin(n\omega x)\sin(m\omega x)$$
$$\cos((n-m)\omega x) = \cos(n\omega x)\cos(m\omega x) + \sin(n\omega x)\sin(m\omega x)$$

Sum of these two equations results in

$$\cos(n\omega x)\cos(m\omega x) = \frac{1}{2}[\cos((n+m)\omega x) + \cos((n-m)\omega x)]$$

Then

$$\int_a^{a+P} \cos(n\omega x)\cos(m\omega x)\,dx = \frac{1}{2}\int_a^{a+P} [\cos((n+m)\omega x) + \cos((n-m)\omega x)]dx = 0$$

Similarly (see Kreyszig, page 479) one can prove that at $n \neq m$

$$\int_a^{a+P} \cos(n\omega x)\sin(m\omega x)\,dx = 0, \qquad \int_a^{a+P} \sin(n\omega x)\sin(m\omega x)\,dx = 0$$

5.4. Fourier and generalized Fourier series

Let's consider the system of functions $f_0(x), f_1(x), f_2(x)$,... which are orthogonal at $[a, b]$ with respect to weight $r(x)$ and assume that some periodic function with period $P = 2\pi/\omega = b - a$ can be represented in the form:

$$f(x) = \sum_{m=0}^{\infty} a_m f_m(x) \qquad (5.4.1)$$

The **coefficients** a_m in Eq. (5.4.1) can be found with the following theorem:

Theorem:

If function $f(x)$ can be represented in the form given by Eq. (5.4.1), then coefficients in this series are unique and can be found with the following **Euler formulas**:

$$a_m = \frac{(f, f_m)}{\|f_m\|^2} = \frac{1}{\|f_m\|^2} \int_a^b r(x) f(x) f_m(x) dx \qquad (5.4.2)$$

Proof:

Let's multiply Eq. (5.4.1) by $r(x) f_n(x)$ and integrate it from a to b:

$$\int_a^b r(x) f(x) f_n(x) dx = \sum_{m=0}^{\infty} a_m \int_a^b r(x) f_n(x) f_m(x) dx$$

or

5.4. Fourier and generalized Fourier series

$$(f, f_n) = \sum_{m=0}^{\infty} a_m (f_n, f_m).$$

Now let's use the orthogonality: $(f_n, f_m) = 0$ if $n \neq m$. Then

$$(f, f_n) = a_n (f_n, f_n) = a_n \|f_n\|^2.$$

The representation of the function $f(x)$ in the form given by Eq. (5.4.1) where coefficients are calculated with Eqs. (5.4.2) is called the **generalized Fourier series**.

Fourier series

Now let's apply our general theorem to the trigonometric system of functions

$$f_0(x), f_1(x), f_2(x): \; 1, \cos \omega x, \sin \omega x, \cos 2\omega x, \sin 2\omega x, \dots, \cos n\omega x, \sin n\omega x, \dots .$$

where $P = 2\pi/\omega = 2L$ is the fundamental period of function $f(x)$.

According to Eq. (5.3.1) $\|f_0\|^2 = P = 2L$, $\|f_m\|^2 = P/2 = L$ at $m > 0$. Then let's re-write Eq. (5.4.1) in the form

$$f(x) = \sum_{m=0}^{\infty} \frac{(f, f_m)}{\|f_m\|^2} f_m(x) = \frac{(f, f_0)}{\|f_0\|^2} + \sum_{m=1}^{\infty} \frac{(f, f_m)}{\|f_m\|^2} f_m(x) = \frac{(f, f_0)}{P} + \sum_{m=1}^{\infty} \frac{(f, f_m)}{P/2} f_m(x)$$

5.4. Fourier and generalized Fourier series

In the sum, let's group together cos and sin functions of the same argument. Then

$$f(x) = \frac{1}{2}\frac{(f,1)}{P/2} + \sum_{n=1}^{\infty}\left[\frac{(f,\cos(n\omega x))}{P/2}\cos(n\omega x) + \frac{(f,\sin(n\omega x))}{P/2}\sin(n\omega x)\right]$$

Now let's introduce the **Fourier coefficients** a_n and b_n of function $f(x)$:

These are Fourier **coefficients** of function $f(x)$

$$a_0 = \frac{(f,1)}{P/2} = \frac{2}{P}\int_a^{a+P} f(x)dx$$

$$a_n = \frac{(f,\cos(n\omega x))}{P/2} = \frac{2}{P}\int_a^{a+P} f(x)\cos(n\omega x)\,dx \qquad n = 1,2\ldots \qquad (5.4.3)$$

$$b_n = \frac{(f,\sin(n\omega x))}{P/2} = \frac{2}{P}\int_a^{a+P} f(x)\sin(n\omega x)\,dx$$

Then

This is the Fourier **series** of function $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(n\omega x) + b_n\sin(n\omega x)] \qquad (5.4.4)$$

The representation of function $f(x)$ in the form (5.4.4) where coefficients are calculated with Eqs. (5.4.3) is called the **Fourier series** of function $f(x)$.

Different forms of the Fourier series

$P = 2\pi/\omega = 2L$

If we use ω:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(n\omega x) + b_n \sin(n\omega x)]$$

$$a_0 = \frac{\omega}{\pi}\int_a^{a+2\pi/\omega} f(x)dx\,,\; a_n = \frac{\omega}{\pi}\int_a^{a+2\pi/\omega} f(x)\cos(n\omega x)\,dx\,,\; b_n = \frac{\omega}{\pi}\int_a^{a+2\pi/\omega} f(x)\sin(n\omega x)\,dx$$

If we use P:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos\frac{2\pi n x}{P} + b_n \sin\frac{2\pi n x}{P}]$$

$$a_0 = \frac{2}{P}\int_a^{a+P} f(x)dx\,, \qquad a_n = \frac{2}{P}\int_a^{a+P} f(x)\cos\frac{2\pi n x}{P}dx\,, \qquad b_n = \frac{2}{P}\int_a^{a+P} f(x)\sin\frac{2\pi n x}{P}dx$$

If we use L:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos\frac{\pi n x}{L} + b_n \sin\frac{\pi n x}{L}]$$

$$a_0 = \frac{1}{L}\int_a^{a+2L} f(x)dx\,, \qquad a_n = \frac{1}{L}\int_a^{a+2L} f(x)\cos\frac{\pi n x}{L}dx\,, \qquad b_n = \frac{1}{L}\int_a^{a+2L} f(x)\sin\frac{\pi n x}{L}dx$$

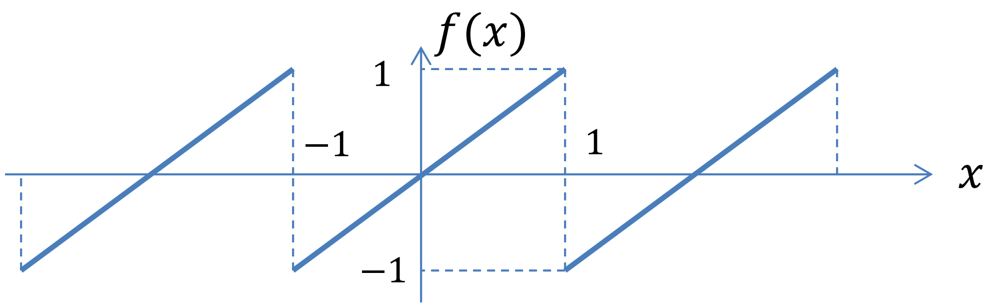
5.4. Fourier and generalized Fourier series

In terms of magnitude and phase:

$$A_n = \sqrt{a_n^2 + b_n^2}, \qquad \varphi_n = \arctan(b_n/a_n)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\omega x - \varphi_n) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \sin(n\omega x + \varphi_n)$$

Example: Let's consider the periodic function given by the plot



$P = 2, L = 1, \omega = \pi$

At period $x \in (-1,1], f(x) = x$

Fourier coefficients according to the Euler formulas (5.3.2)

$$a_0 = \int_{-1}^{1} x dx = 0, a_n = \int_{-1}^{1} x \cos(n\pi x)\, dx = 0$$

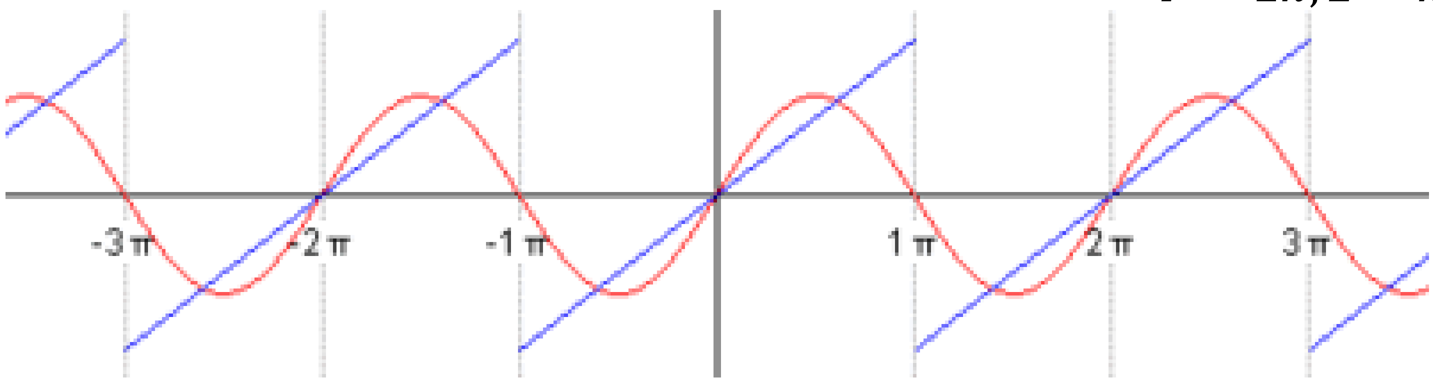
$$b_n = \int_{-1}^{1} x \sin(n\pi x)\, dx = -\frac{1}{\pi n} \int_{-1}^{1} x\, d\cos(n\pi x) = -\frac{1}{\pi n}\left[x\cos(n\pi x)\Big|_{x=-1}^{x=+1} - \int_{-1}^{1} \cos(n\pi x) dx\right]$$

$$= -\frac{2}{\pi n}\cos(n\pi) = \frac{2(-1)^{n+1}}{\pi n}$$

5.4. Fourier and generalized Fourier series

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\pi n} \sin(n\pi x)$$

$P = 2\pi, L = \pi, \omega = 1$



We have obtained *good approximation of a function with discontinuities valid at any* x*!*

What if we will try to use the Taylor expansion?

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 +$$

In the point $a = 0$ we have

$$f(x) = x$$

This is great (accurate results for our $f(x)$) inside a period, but becomes meaningless outside the period if the function is discontinuous.

Conclusion: The Fourier expansion provides good approximation ax arbitrary x of almost any discontinuous (but periodic functions), while the Taylor expansion can be applied only inside intervals where the function is continuous and has all continuous derivatives.

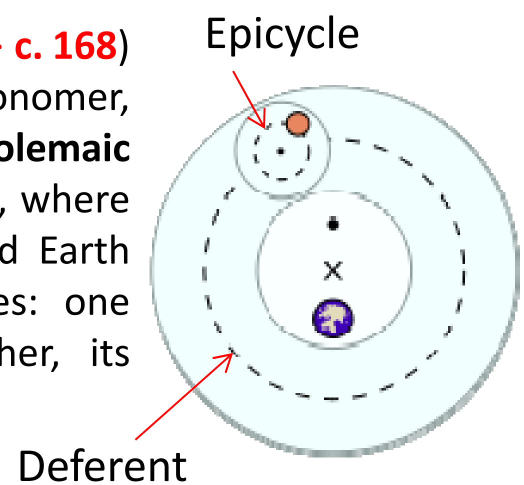
5.4. Fourier and generalized Fourier series

Jean Baptiste Joseph Fourier (21 March 1768 – 16 May 1830) was a French mathematician and physicist best known for initiating the investigation of **Fourier series** and their applications to problems of heat transfer and vibrations. The **Fourier transform** and **Fourier's Law** are also named in his honor.

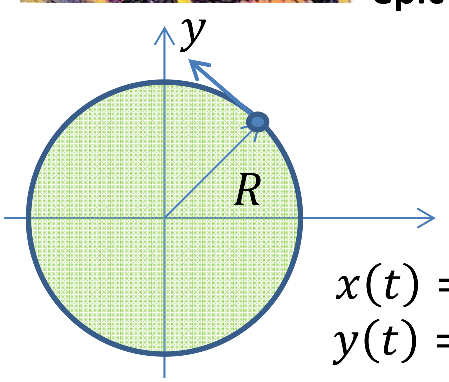
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

But was Joseph Fourier the first who "invented" the Fourier Series?

Claudius Ptolemy (c. AD 90 – c. 168) was a Greco-Egyptian astronomer, who invented the **Ptolemaic geocentric model** of universe, where each planet is moved around Earth by a system of two spheres: one called its **deferent**, the other, its **epicycle.**



Imperfections in the Ptolemaic system were discovered through observations accumulated over time. More levels of epicycles (circles within circles) can be added to the model to match more accurately the observed planetary motions.



$$x(t) = R\cos\omega t$$
$$y(t) = R\sin\omega t$$

- Cosine and sine functions parametrically define the circular motion.
- Representation of the visible trajectory of a planet in the form the system of deferent + epicycles is equivalent to the expansion of the trajectory into the Fourier series.

5.5. Fourier expansions for functions satisfying the Dirichlet conditions

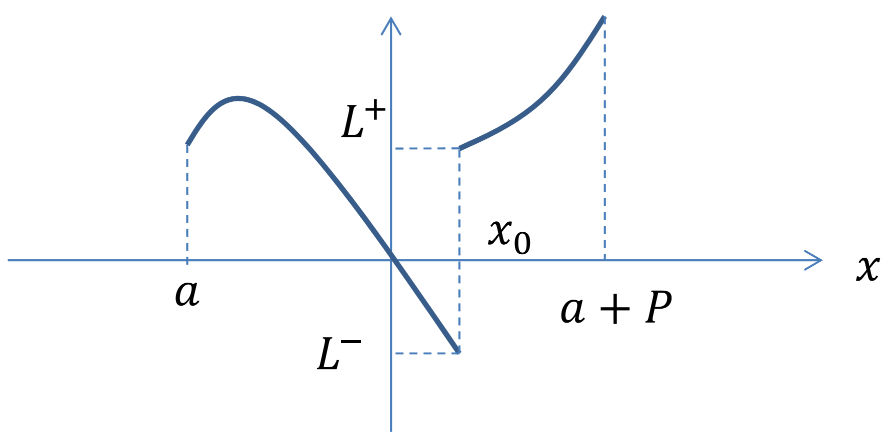
Our goal is to formulate conditions for a function $f(x)$ which guarantee that the Fourier series for this function exists and converges to the values of the function. Such conditions are known as the Dirichlet conditions.

Let's consider some periodic (real value) function $f(x)$ with period P. We say that this function satisfies the **Dirichlet conditions** if

1. This function is a piecewise monotonous function in any interval $[a, a+P]$ (This means that $f(x)$ has a finite number of extrema in this interval).

2. This function is a piecewise continuous function in any interval $[a, a+P]$ (This means that $f(x)$ has a finite number of discontinuities in this interval).

3. This function has finite limits in the ends of the interval $[a, a+P]$ and finite left- and right-hand limits at any discontinuity (i.e. all discontinuities are **jump** or **step** discontinuities).



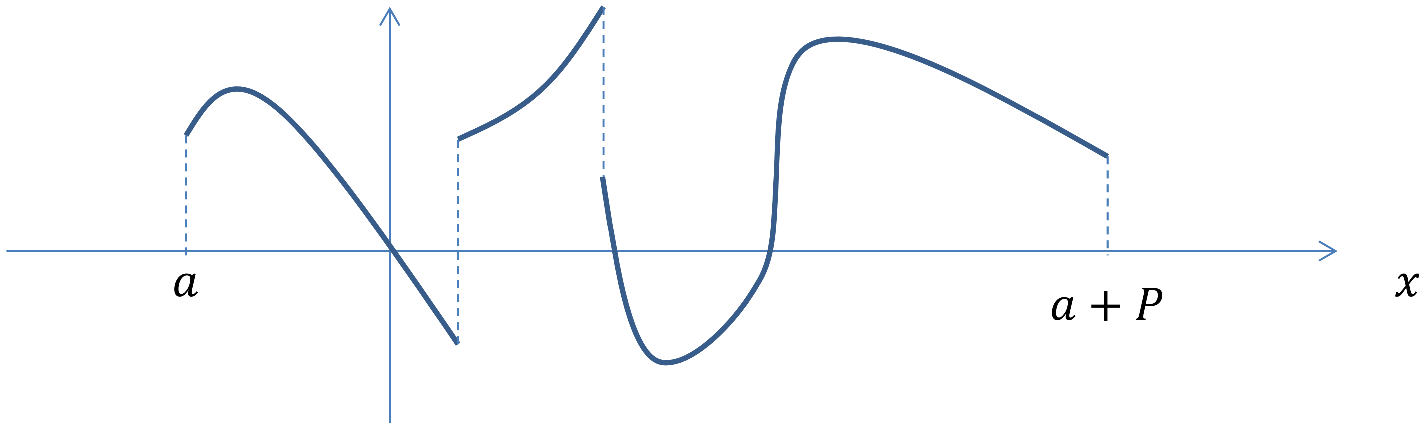
Left-hand limit :

$$L^- = f(x_0 - 0) = \lim_{h>0, h\to 0} f(x_0 - h)$$

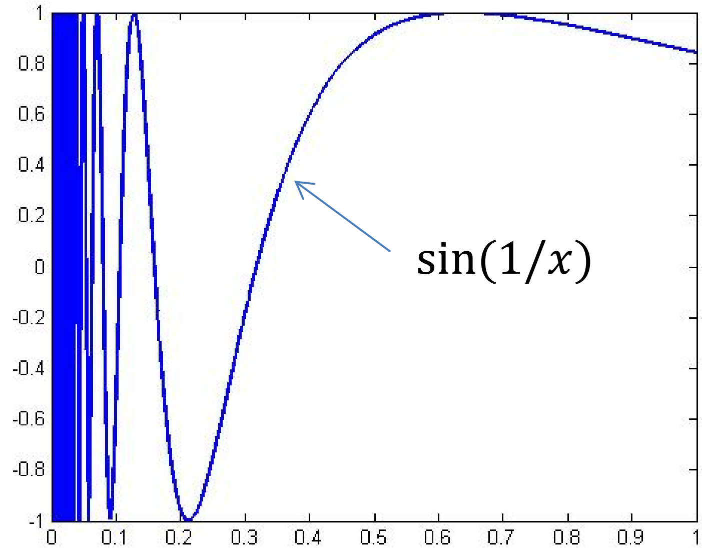
Right-hand limit :

$$L^+ = f(x_0 + 0) = \lim_{h>0, h\to 0} f(x_0 + h)$$

Example 1: A function satisfying the Dirichlet conditions



Example 2: Function $\sin(1/x)$ does not satisfy the Dirichlet conditions, since it is not a piecewise monotonous function in any interval containing 0.



5.5. Fourier expansions for functions satisfying the Dirichlet conditions

Theorem: **Dirichlet theorem**

If $f(x)$ is a periodic function with the fundamental period $P = 2\pi/\omega$ and satisfies the Dirichlet conditions, then

1. The Fourier series

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega x) + b_n \sin(n\omega x)]$$

where the Fourier coefficients are calculated with the Euler formulas (2.3.4) converge at any x.

2. $S(x)$ is also a periodic function with the fundamental period P.

3. If $f(x)$ is continuous in the point x, then

$$S(x) = f(x)$$

4. If $f(x)$ is discontinuous in the point x, then values of $S(x)$ is the half-sum of left- and right-hand limits of $f(x)$:

$$S(x) = \frac{1}{2}[f(x-0) + f(x+0)]$$

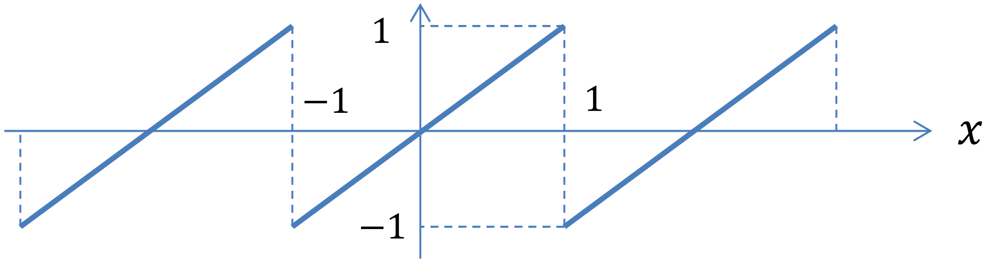
5. The Fourier coefficients approach 0 when $n \to \infty$: $a_n, b_n \to 0$ if $n \to \infty$.

6. The "speed" of convergence of a_n and b_n to zero with increasing n depends on the degree of smoothness of $f(x)$: If $f(x)$ has discontinuous derivatives of order k, then

$$A_n = \sqrt{a_n^2 + b_n^2} = O\left(\frac{1}{n^{k+1}}\right)$$

5.5. Fourier expansions for functions satisfying the Dirichlet conditions

Example:

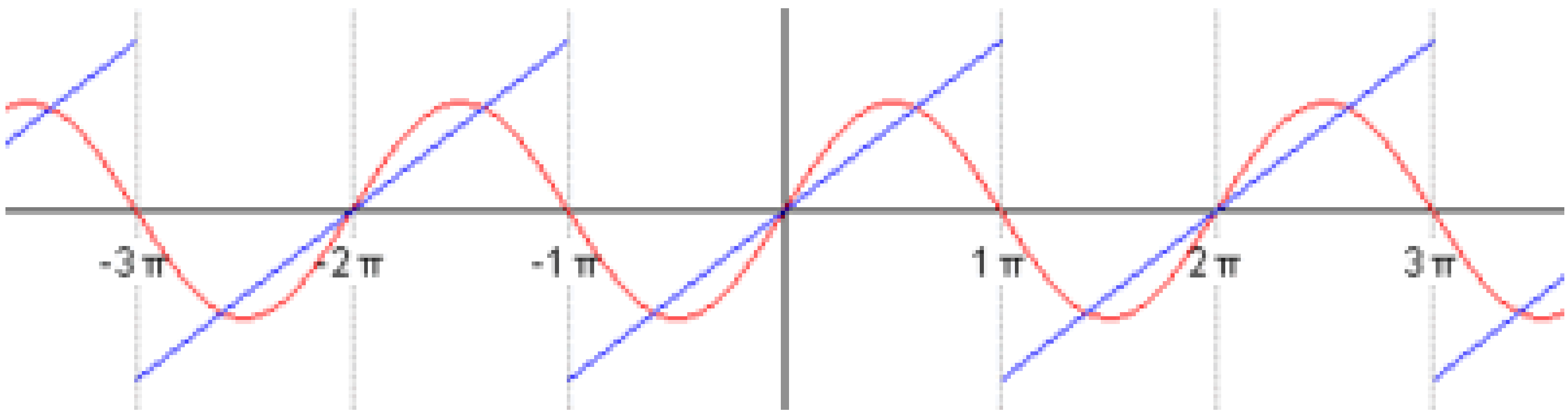


$$P = 2, L = 1, \omega = \pi$$

At period $x \in (-1,1], f(x) = x$

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\pi n} \sin(n\pi x)$$

$$P = 2\pi, L = \pi, \omega = 1$$



We see that:

$$S(x) = \frac{1}{2}[f(x-0) + f(x+0)], \qquad a_n = O\left(\frac{1}{n}\right)$$

5.6. Complex Fourier series

Let's assume that some P-periodic function $f(x)$ can be represented in the form of the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega x) + b_n \sin(n\omega x)] \qquad (5.6.1)$$

$$a_0 = \frac{2}{P}\int_a^{a+P} f(x)dx\,, a_n = \frac{2}{P}\int_a^{a+P} f(x)\cos(n\omega x)\,dx\,, \quad b_n = \frac{2}{P}\int_a^{a+P} f(x)\sin(n\omega x)\,dx\,, n = 1,2\,...$$

Let's show that every $a_n \cos(n\omega x) + b_n \sin(n\omega x)$ can be represented in a form containing complex numbers using the **Euler formula** for the complex exponent:

$$\boxed{e^{i\varphi} = \cos\varphi + i\sin\varphi} \qquad (5.6.2)$$

Then

$$\cos(n\omega x) = \frac{e^{in\omega x} + e^{-in\omega x}}{2}, \qquad \sin(n\omega x) = -i\frac{e^{in\omega x} - e^{-in\omega x}}{2}$$

$$a_n \cos(n\omega x) + b_n \sin(n\omega x) = a_n \frac{e^{in\omega x} + e^{-in\omega x}}{2} - ib_n \frac{e^{in\omega x} - e^{-in\omega x}}{2} =$$

$$\frac{a_n - ib_n}{2} e^{in\omega x} + \frac{a_n + ib_n}{2} e^{-in\omega x} \qquad (5.6.3)$$

5.6. Complex Fourier series

Now let's introduce the **complex Fourier amplitudes** c_k $(-\infty < k < \infty)$:

$$c_n = \frac{a_n - ib_n}{2} = \frac{1}{P}\int_a^{a+P} f(x)[\cos(n\omega x) - i\sin(n\omega x)]dx = \frac{1}{P}\int_a^{a+P} f(x)e^{-in\omega x}dx$$

$$c_{-n} = \frac{a_n + ib_n}{2} = \frac{1}{P}\int_a^{a+P} f(x)[\cos(n\omega x) + i\sin(n\omega x)]dx = \frac{1}{P}\int_a^{a+P} f(x)e^{in\omega x}dx$$

$$c_0 = \frac{a_0}{2}$$

Now let's insert (5.6.3) into Eq. (5.6.1). Then we obtain:

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega x}, \qquad c_k = \frac{1}{P}\int_a^{a+P} f(x)e^{-ik\omega x}dx \qquad (5.6.4)$$

Eq. (5.6.4) is the **complex Fourier series** of the P-periodic real-valued function $f(x)$.

If the complex Fourier amplitudes are found, then :

$$a_n = 2\,Re\,c_n = 2\,Re\,c_{-n} = Re\,(c_{-n} + c_n)$$

$$b_n = -\,2\,Im\,c_n = 2\,Im\,c_{-n} = Im\,(c_{-n} - c_n)$$

5.7. Fourier series of even and odd periodic functions

Function $f(x)$ is called the **odd function** if

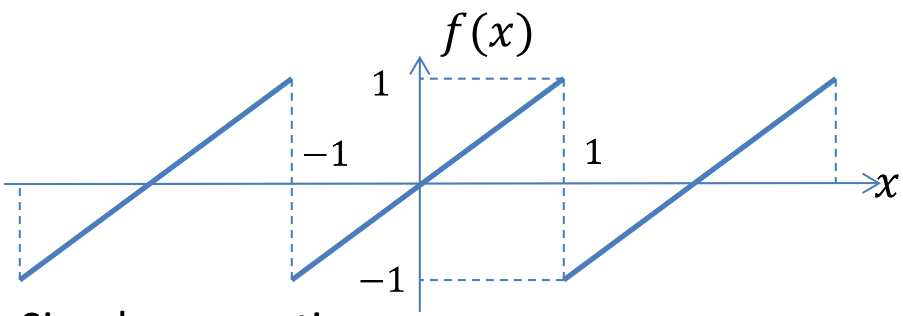
$$f(-x) = -f(x)$$

Function $f(x)$ is called the **even function** if

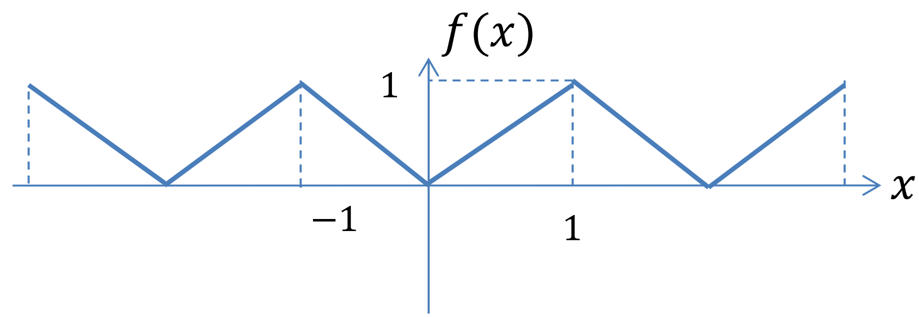
$$f(-x) = f(x).$$

Examples:

Odd functions: $\sin x$,



Even functions: $\cos x$,



Simple properties:

1. If $f(x)$ and $g(x)$ are even functions and $p(x)$ and $q(x)$ are odd functions, then $f(x)g(x)$ and $p(x)q(x)$ are even functions, $f(x)p(x)$ is the odd function.

2. If $p(x)$ is an odd function, then integral over any interval symmetric with respect to 0 is equal to zero

Variable change, $y = -x$, in this integral

$$\int_{-a}^{a} p(x)dx = \int_{-a}^{0} p(x)dx - \int_{0}^{a} p(-x)dx = \int_{-a}^{0} p(x)dx + \int_{0}^{-a} p(y)dy = 0$$

5.7. Fourier series of even and odd periodic functions

Fourier expansion of an even $\boldsymbol{P}$-periodic function $\boldsymbol{f(x)}$:

Let's take $a = -P/2 = -L$:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx\,, \qquad a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos(n\omega x)}_{\text{Even}}\,dx\,, \quad b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin(n\omega x)}_{\text{Odd}}\,dx = 0,$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega x)$$

All c_k are purely real

For an even function, phase $\varphi_n = \arctan(b_n/a_n) = 0$; In the complex Fourier series $c_n = c_{-n}$.

Fourier expansion of an odd $\boldsymbol{P}$-periodic function $\boldsymbol{f(x)}$:

Let's take $a = -P/2 = -L$:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx\,, = 0 \quad a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos(n\omega x)}_{\text{Odd}}\,dx = 0\,, \quad b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin(n\omega x)}_{\text{Even}}\,dx\,,$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\omega x)$$

All c_k $(k > 0)$ are purely imaginary

For an odd function, phase $\varphi_n = \arctan(b_n/a_n) = \frac{\pi}{2}$; In the complex Fourier series $c_n = -c_{-n}$.

5.7. Fourier series of even and odd periodic functions

Even/Odd decomposition of a P-periodic function $f(x)$:

Any function can be uniquely decomposed into a sum of even, $f_e(x)$, and odd, $f_o(x)$, functions:

$$f(x) = f_e(x) + f_o(x)$$

where

$$f_e(x) = \frac{1}{2}[f(x) + f(-x)], \qquad f_o(x) = \frac{1}{2}[f(x) - f(-x)]$$

It allows one to introduce another form of the Fourier series:

$$f_e(x) = \frac{a_{0(e)}}{2} + \sum_{n=1}^{\infty} a_{n(e)} \cos(n\omega x)$$

$$f_o(x) = \sum_{n=1}^{\infty} b_{n(o)} \sin(n\omega x)$$

$$f(x) = \frac{a_{0(e)}}{2} + \sum_{n=1}^{\infty} [a_{n(e)} \cos(n\omega x) + b_{n(o)} \sin(n\omega x)]$$

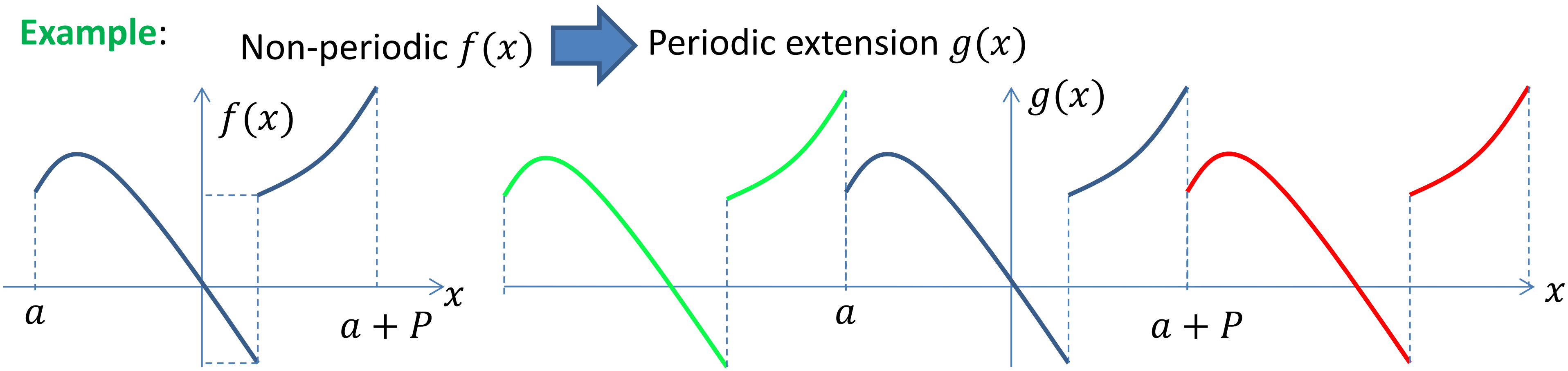
where

$$a_{0(e)} = \frac{1}{L} \int_{-L}^{L} f_e(x) dx\,, \qquad a_{n(e)} = \frac{1}{L} \int_{a}^{a+P} f_e(x) \cos(n\omega x)\, dx\,, \quad b_{n(o)} = \frac{1}{L} \int_{a}^{a+P} f_o(x) \sin(n\omega x)\, dx$$

5.8. Fourier series of non-periodic function given at finite interval. Half-range series

Let's consider a non-periodic function $f(x)$ given at the finite interval $x \in [a, b]$ ($|b - a| < \infty$) and assume that $f(x)$ satisfies the Dirichlet conditions.

Example:



Although $f(x)$ is non-periodic, it can be expanded into the Fourier series. For this purpose we need to periodically extend $f(x)$ for arbitrary x. For instance, we can introduce a new periodic function $g(x)$ with the period $P = b - a$ which is defined as follows:

> This formal definition of $g(x)$ is not necessary for calculations of a_n and b_n, because only values of $f(x)$ at $a \le x \le a + P$ will be involved

$$g(x) = \begin{cases} f\left(x - P\left[\frac{x-a}{P}\right]\right) & x > a \\ f\left(x + P + P\left[\frac{x-a}{P}\right]\right) & x < a \end{cases} \qquad (5.8.1)$$

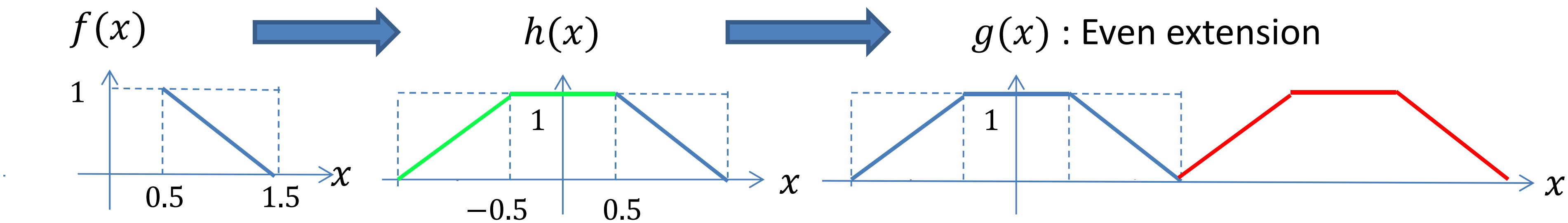
where $[a]$ is the integer part of the real number a.

The **periodic extension** $g(x)$ satisfies the Dirichlet conditions and, thus, can be represented in the form of the Fourier series. In the interval $x \in [a, b]$ the Fourier expansion of $g(x)$ will coincides with $f(x)$ in all points except the discontinuities of $g(x)$.

5.8. Fourier series of non-periodic function given at finite interval. Half-range series

For any non-periodic $f(x)$ given in a finite interval, there is infinitely large number of different periodic extensions with different fundamental periods $P \geq b - a$.

Example:



In particular, for any $f(x)$ given at $x \in [a, b]$ one can introduce an extension $g(x)$ which can be either even or odd periodic function of the period $P \geq 2(b - a)$. The Fourier series for $f(x)$ obtained with the help of even or odd periodic functions are called **half-range Fourier expansions** or **series**.

Let's first consider the case $a = 0$. Then we can introduce the even or odd periodic extension in two steps:

1. First, we introduce an auxiliary function $h(x)$ given at $-L < x < L$, $L = b - a$.

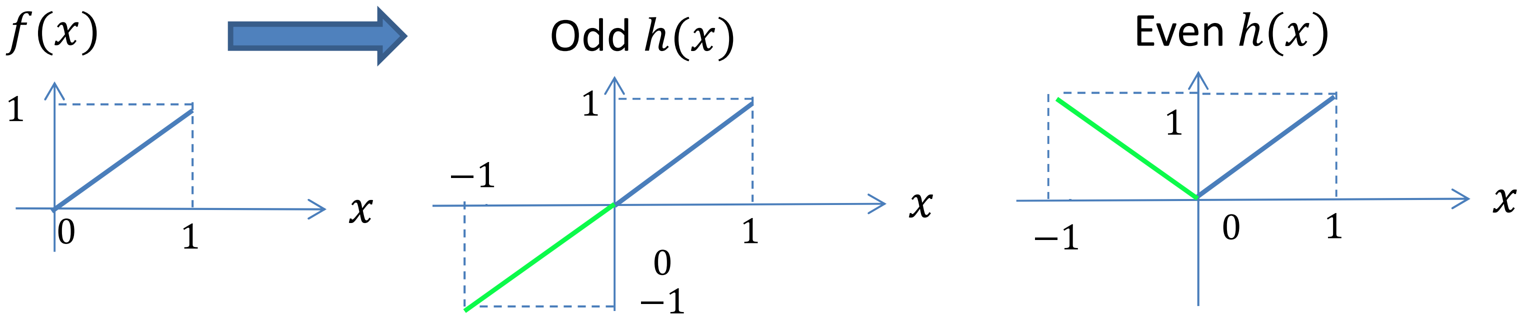
a. For the odd extension: $h(x) = \begin{cases} f(x) & x > 0 \\ -f(-x) & x < 0 \end{cases}$

b. For the even extension: $h(x) = \begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases}$

2. Second, we introduce a periodic expansion $g(x)$ for $h(x)$ given by Eq. (5.8.1).

5.8. Fourier series of non-periodic function given at finite interval. Half-range series

Example:



If $a \neq 0$, then the odd or even extension can be introduced by two ways:

1. If $a > 0$ or $b < 0$ we can introduce an extended function in a way illustrated in the figure in the previous slide.

2. In the general case, we can introduce a shifted function $\tilde{f}(x) = f(x-a)$, which is defined in $0 < x < L = b - a$ and then apply our two-step algorithm from the previous slide to $\tilde{f}(x)$.

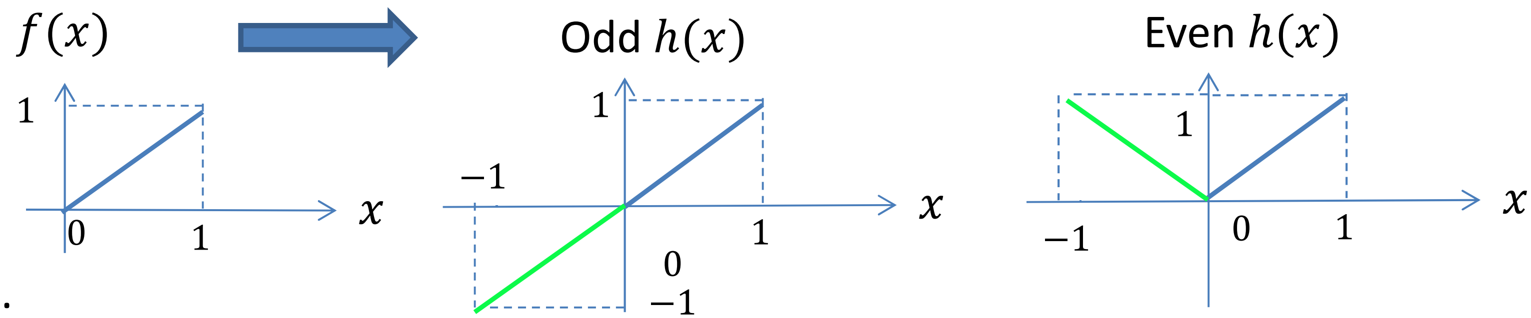
The half-range Fourier expansions are convenient to use, since only *half of all Fourier coefficients should be determined*: $b_n = 0$ for even periodic extensions, $a_n = 0$ for odd periodic extensions.

Question: Which extension is better?

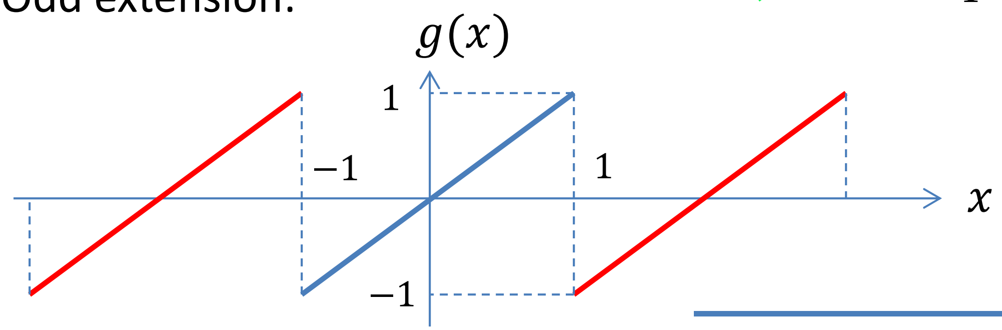
In general, the better results are obtained with an extension which removes the discontinuities in $g(x)$. Then, according to the Dirichlet theorem, the convergence of a_n and b_n to zero with increasing n is faster and we can retain smaller number of terms in the Fourier sum in practical calculations.

5.8. Fourier series of non-periodic function given at finite interval. Half-range series

Example:



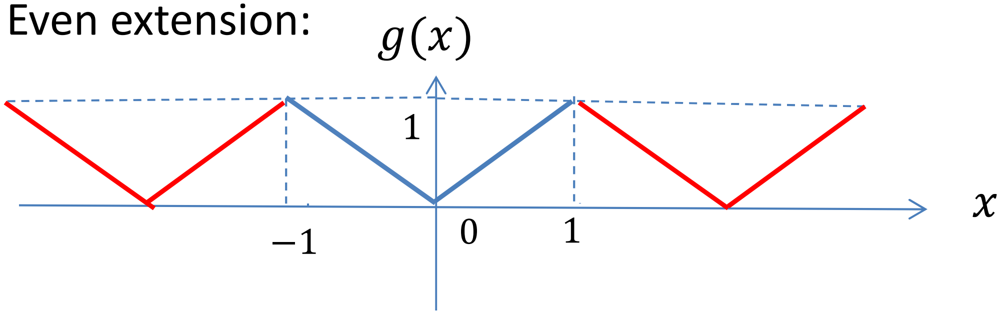
Odd extension:



See sect. 5.4, Slide 14:

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\pi n} \sin(n\pi x)$$

Even extension:



$$f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2[(-1)^n - 1]}{(\pi n)^2} \cos(n\pi x)$$

$$a_0 = 2\int_0^1 x dx = 1$$

$$a_n = 2\int_0^1 x\cos(n\pi x)\,dx = \frac{2}{\pi n}\left[x\sin(n\pi x)\Big|_{x=0}^{x=1} - \int_0^1 \sin(n\pi x)dx\right] = \frac{2[(-1)^n - 1]}{(\pi n)^2}$$

In this case the *even extension is better*, since it removes discontinuities and provides $a_n \sim 1/n^2$.

5.9. Parseval's identity

Many applications of the Fourier series use the following theorem:

Theorem: **Parseval's theorem**

Let's consider a P-periodic function $f(x)$ that satisfies the Dirichlet conditions and, thus can be represented in the form of the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega x) + b_n \sin(n\omega x)] \tag{5.9.1}$$

Then **Parseval's identity** holds:

$$\frac{1}{P} \int_{a}^{a+P} f^2(x) dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \tag{5.9.2}$$

Proof:

Let's rewrite Eq. (5.9.1) using the following notation for the functions of the trigonometric system: $f_0(x) = 1, f_1(x) = \cos \omega x$, $f_2(x) = \sin \omega x$, etc., and $d_0 = a_0/2$, $d_1 = a_1$, $d_2 = b_1$, etc.

$$f(x) = \sum_{n=0}^{\infty} d_n f_n(x)$$

And then let's calculate the product

$$f^2(x) = \left(\sum_{n=0}^{\infty} d_n f_n(x)\right)\left(\sum_{m=0}^{\infty} d_m f_m(x)\right) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} d_n f_n(x) d_m f_m(x) \tag{5.9.3}$$

5.9. Parseval's identity

Now we can integrate Eq. (5.9.3) over a period:

$$\int_a^{a+P} f^2(x)dx = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} d_n d_m \int_a^{a+P} f_n(x) f_m(x)dx$$

But the trigonometric system of functions is orthogonal with respect to the weight $r(x) = 1$, i.e.

$$\int_a^{a+P} f_n(x) f_m(x)dx = \begin{cases} 0 & n \neq m \\ P & n = m = 0 \\ P/2 & n = m > 0 \end{cases}$$

$$\int_a^{a+P} f^2(x)dx = P d_0^2 + \frac{P}{2}\sum_{n=1}^{\infty} d_n^2$$

Consequence: If we use the complex Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega x}$$

Then Parseval's identity takes the form

$$\frac{1}{P}\int_a^{a+P} f^2(x)dx = \sum_{k=-\infty}^{\infty} |c_k|^2 \qquad (5.9.4)$$

5.9. Parseval's identity

In order to prove it, let's use the definition of c_k:

$$c_n = \frac{a_n - ib_n}{2}, \qquad c_{-n} = \frac{a_n + ib_n}{2}, \qquad c_0 = \frac{a_0}{2}$$

Then $|c_n|^2 = |c_{-n}|^2 = (a_n^2 + b_n^2)/4$

$$\sum_{k=-\infty}^{\infty} |c_k|^2 = |c_0|^2 + \sum_{k=-\infty,\, k\neq 0}^{\infty} |c_k|^2 = \left(\frac{a_0}{2}\right)^2 + 2\sum_{k=1}^{\infty} |c_k|^2 = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

Energy spectrum

Many applications of Parseval's identity are based on the interpretation of $(a_n^2 + b_n^2)/2$ as energy (or power) associated with a particular term/oscillation in the Fourier series.

Example 1: Mechanical oscillations.
Assume that $y(t)$ describes displacement of an oscillating mass in the mass-spring system.
Let's first consider a harmonic oscillation of an undamped system with the equation of motion

$$y_n'' + \omega_n^2 y = 0$$

The solution is $y_n(t) = a_n \cos(\omega_n t) + b_n \sin(\omega_n t)$ (here $\omega_n = n\omega$) and the energy "stored" in this harmonic oscillation is equal to ($\omega_n^2 = k/m_n$, $y'_n(t) = \omega_n[-a_n \sin(\omega_n t) + b_n \cos(\omega_n t)]$)

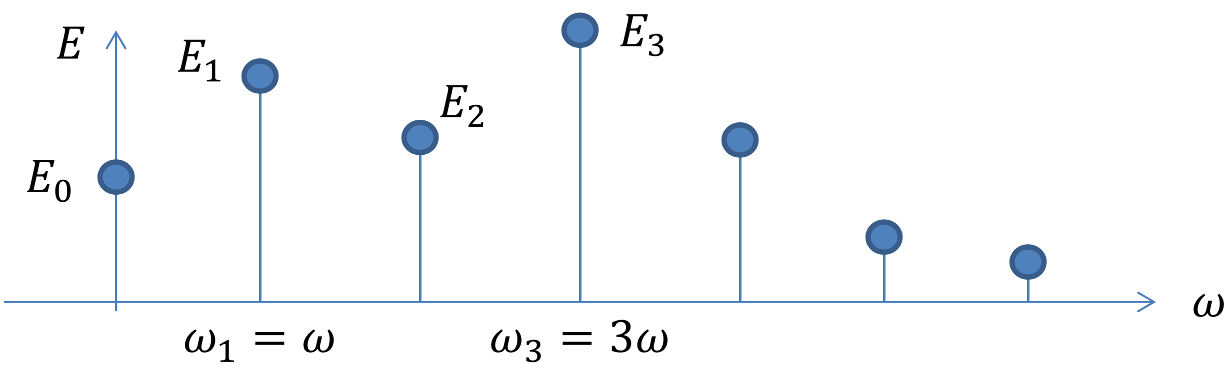
$$E_n = E_{kinetic} + E_{potential} = \frac{1}{2} m_n (y_n')^2 + \frac{1}{2} k (y_n)^2 = \frac{k}{2}\left[\frac{(y_n')^2}{\omega_n^2} + (y_n)^2\right] = k\,\frac{a_n^2 + b_n^2}{2}$$

5.9. Parseval's identity

Thus, if we have a non-harmonic oscillation $y(t)$ that can be represented as a superposition of infinite number of harmonic oscillations (i.e. in the form of the Fourier series), the averaged over a period energy stored in the oscillation $y(t)$ is a sum of energies stored in the individual harmonic oscillations (in every harmonic oscillation $E_n = const$ and does not depend on time):

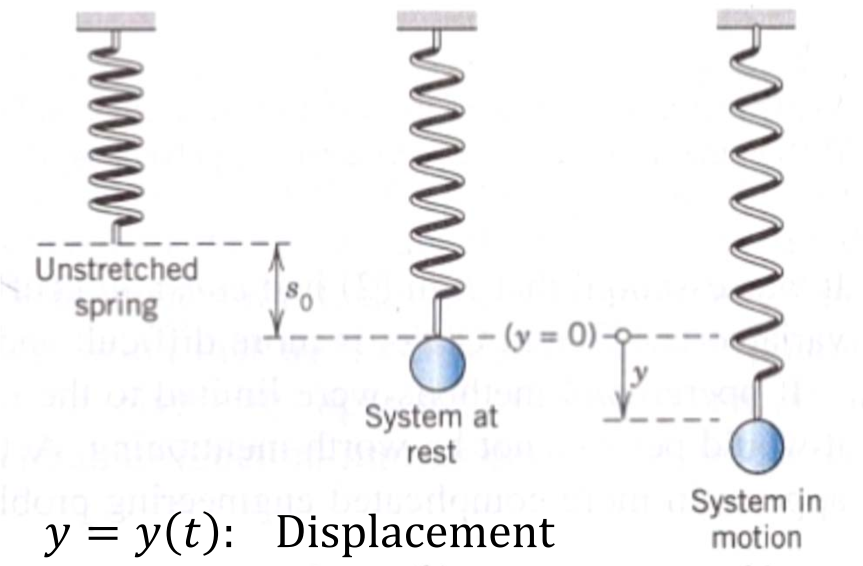
$$\frac{E}{k} = \sum_{n=0}^{\infty} \frac{E_n}{k} = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{P}\int_a^{a+P} y^2(t)dt$$

Distribution of averaged energy over different oscillation frequencies is called the **energy** (**power**) **spectrum** of oscillation $y(t)$. In this regard, we say that any periodic function (oscillation) has a **discrete** or **point** spectrum, since *the energy of such oscillation is stored in countably many isolated frequencies* ω_n.

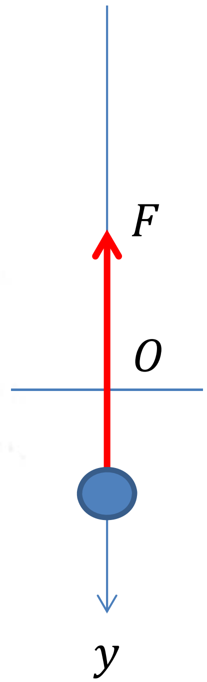


Example 2: Joule heat in electrical circuits. Let's consider a part of an electrical circuit with resistance R and current $I(t)$. Then the Joule heat dissipated during time dt is equal to $dQ = RI^2dt$. If we would calculate the Fourier coefficients for $I(t)$, then value $(a_n^2 + b_n^2)/2$ is proportional to the contribution of oscillation of frequency $n\omega$ to the total electric power RI^2.

Mechanical mass-spring system



$y = y(t)$:   Displacement



Newton's second law of motion:

$$my'' = \sum F_i = F_1 + F_2 + F_3$$

1. **Elastic restoring force** (Hook's law):

$$F_1 = -ky$$

k is the **spring constant** (spring stiffness)

2. **Damping (friction) force**:

$$F_2 = -cy'$$

3. **Input (driving) external force**

$$F_3 = r(t)$$

$$\boxed{my'' + cy' + ky = r(t)} \qquad (5.10.1)$$

Undamped oscillation : $c = 0$
Damped oscillation: $c \neq 0$

Free oscillation: $r(t) = 0$
Forced oscillation: $r(t) \neq 0$

Solution for harmonic driving force (See Section 2.8)

We considered only the **harmonic driving force**, $r(t) = F_0 \cos \omega\, t$, where ω is the **input angular frequency**

$$my'' + cy' + ky = F_0 \cos \omega\, t$$

$$y(t) = y_h(t) + y_p(t)$$

In the case of damped oscillation ($c > 0$), $y_h(t) \to 0$ when $t \to \infty$, so that (see Section 2.8)

$$y(t) \to y_p(t)$$

$$y_p(t) = A \cos \omega t + B \sin \omega t \qquad (5.10.2)$$

$$A = F_0 \frac{m\left({\omega_0}^2 - \omega^2\right)}{\left({\omega_0}^2 - \omega^2\right)^2 m^2 + c^2\omega^2}, \qquad B = F_0 \frac{\omega c}{\left({\omega_0}^2 - \omega^2\right)^2 m^2 + c^2\omega^2}, \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

Using the Fourier series, we can generalize the solution for an arbitrary periodic driving force $r(t)$. Let's assume that

- $r(t)$ is the P-periodic function which satisfies the Dirichlet conditions, and thus, can be expanded into the Fourier series.
- Mean value of $r(t)$ is zero: $\frac{1}{P}\int_a^{a+P} r(x)dx = \frac{a_0}{2} = 0$.
- $r(t)$ is the even function (only for the sake of simplicity, the general case can be considered).

Then ($P = 2L$):

$$r(t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L}, \qquad a_n = \frac{1}{L}\int_{-L}^{L} r(t)\cos\frac{n\pi t}{L}\,dt$$

From Eq. (5.10.1):

$$my'' + cy' + ky = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} \qquad (5.10.3)$$

Let's consider an equation for a single Fourier term n ($\omega_n = n\pi/L$):

$$my''_{(n)} + cy'_{(n)} + ky_{(n)} = a_n \cos\omega_n t \qquad (5.10.4)$$

The particular solution of the non-homogeneous ODE (5.10.4) is given by Eq. (5.10.2):

$$y_{p(n)}(t) = A_n \cos\omega_n t + B_n \sin\omega_n t$$

$$A_n = a_n \frac{m\left({\omega_0}^2 - \omega_n^2\right)}{\left({\omega_0}^2 - \omega_n^2\right)^2 m^2 + c^2\omega_n^2}, \qquad B_n = a_n \frac{\omega_n c}{\left({\omega_0}^2 - \omega_n^2\right)^2 m^2 + c^2\omega_n^2}$$

Since Eq. (5.10.3) is linear, the particular solution of this equation takes the form:

$$y_p(t) = \sum_{n=1}^{\infty} y_{p(n)}(t) = \sum_{n=1}^{\infty}\left(A_n \cos\omega_n t + B_n \sin\omega_n t\ \right) \qquad (5.10.5)$$

In order to check that Eq. (5.10.5) is a solution of (5.10.3): Substitute Eq. (5.10.5) into (5.10.3).

5.11. Application of the generalized Fourier series: The Sturm-Liouville problem. Solving the heat conduction equation by the separation of variables

The boundary value problem for the *second-order linear ODE*:

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0, \qquad a \le x \le b, \qquad p(a), p(b) \ne 0 \qquad (5.11.1a)$$

$$k_1 y(a) + k_2 y'(a) = 0 \qquad (5.11.1b)$$

$$l_1 y(b) + l_2 y'(b) = 0 \qquad (5.11.1c)$$

where k_1, k_2, l_1, and l_2 are real numbers, is called the **Sturm-Liouville problem**.

Note: The Sturm-Liouville problem is important for solving PDEs with separating variables (example will be considered below).

Any Sturm-Liouville problem has the **trivial** solution $y \equiv 0$ (can be proved by substitution). The fundamental property of the Sturm-Liouville problem, however, is a non-uniqueness of solution at some particular values of λ: At certain λ, other, non-trivial ($y \not\equiv 0$) solutions also exist.

If the Sturm-Liouville problem has a non-trivial solution $y(x)$ at some λ, such a λ is called the **eigenvalue** of the Sturm-Liouville problem, and $y(x)$ is called the **eigenfunction** corresponding to the eigenvalue λ.

To solve the Sturm-Liouville problem means to find all pairs of eigenvalues and eigenfunctions.

Example: $p(x) = 1$, $q(x) = 0$, $r(x) = 1$, $a = 0$, $b = \pi$, $k_1 = 1$, $k_2 = 0$, $l_1 = 1$, and $l_2 = 0$

$$y'' + \lambda y = 0, \qquad y(0) = 0, \qquad y(\pi) = 0 \qquad (5.11.2)$$

If $\lambda = -\nu^2 < 0$, then $y(x) = c_1 e^{\nu x} + c_2 e^{-\nu x}$ and only the trivial solution ($c_1 = c_2 = 0$) satisfies the b.c. (boundary conditions) in Eq. (5.11.2)

If $\lambda = \nu^2 > 0$, then $y(t) = A \cos \nu x + B \sin \nu x$ and it can satisfy the b.c. in Eq. (5.11.2) if

$$A = 0, \qquad \nu\pi = n\pi, \qquad n = 0, \pm 1, \pm 2, \dots$$

but $n = 0$ gives us again the trivial solution and should be excluded.

Thus, the pairs of eigenvalues and eigenfunctions for the problem (5.11.2) are

$$\lambda_n = n^2, \qquad y_n(t) = B \sin nx\,, \qquad n = 1,2,3, \dots \tag{5.11.3}$$

Note: Coefficient B in the eigenfunction is an arbitrary non-zero value. Thus, every eigenfunction for a given λ is non-unique (This is similar to the non-uniqueness of eigenvectors for a given eigenvalue in the matrix eigenvalue problem).

The Sturm-Liouville problem is closely related to the generalized Fourier expansions, since eigenfunctions, corresponding to different λ, form orthogonal systems of functions with respect to the weight $r(x)$ as stated by the following theorem:

Theorem:
Let's assume that $p(x)$, $q(x)$, $r(x)$, and $p'(x)$ in the Sturm-Liouville problem (5.11.1) are real-valued and continuous and $r(x) > 0$ in the interval $a \le x \le b$.
Let $y_m(x)$ and $y_n(x)$ be eigenfunctions that correspond to different eigenvalues λ_m and λ_n ($\lambda_m \ne \lambda_n$).
Then $y_m(x)$ and $y_n(x)$ are orthogonal on $a \le x \le b$ with respect to the weight $r(x)$, i.e.

$$(y_n, y_m) = \int_a^b r(x) y_n(x) y_m(x) dx = \begin{cases} 0, & n \ne m \\ \|y_m\|^2, & n = m \end{cases} \tag{5.11.4}$$

Proof: See Kreyszig, p. 502.

if $p(a) = 0$, then the following **singular** Sturm-Liouville problem can be solved:

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0, \qquad a \le x \le b$$
$$l_1 y(b) + k_2 y'(b) = 0 \tag{5.11.5}$$

If $p(b) = 0$, then the following **singular** Sturm-Liouville problem can be solved:

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0, \qquad a \le x \le b$$
$$k_1 y(a) + k_2 y'(a) = 0 \tag{5.11.6}$$

if $p(a) = p(b)$, then the following Sturm-Liouville problem **with periodic boundary conditions** can be solved:

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0, \qquad a \le x \le b \tag{5.11.7}$$
$$y(a) = y(b), \qquad y'(a) = y'(b)$$

Note: Eigenfunctions of problems (5.11.5)-(5.11.7) also form orthogonal systems of functions (see the proof in Kreyszig, p. 502).

Application of the Fourier expansions for solving PDEs with separating variables

Example: Let's consider the one-dimensional unsteady heat conduction problem:

$$\rho c \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \tag{5.11.8}$$

$$T(0, x) = \tilde{T}_0(x): \quad \text{Initial conditions} \tag{5.11.9}$$

$$dT/dx\Big|_{x=0} = dT/dx\Big|_{x=L} = 0: \quad \text{Boundary conditions (thermally insulated boundaries)} \tag{5.11.10}$$

5.11. Application of the generalized Fourier series: The Sturm-Liouville problem. Solving the heat conduction equation by the separation of variables

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \qquad \alpha = \frac{\kappa}{\rho c} \tag{5.11.11}$$

Let's try to represent the solution in the form

$$T(t,x) = \Theta(t)X(x) \tag{5.11.12}$$

and substitute it into Eq. (5.11.11). Then we obtain

$$\Theta' X = \alpha \Theta X''$$

or

$$\frac{\Theta'}{\Theta} = \alpha \frac{X''}{X}$$

The LHS depends only on t, the RHS depends only on x. The identity is possible only if

$$\frac{\Theta'}{\Theta} = -C = const, \qquad \alpha \frac{X''}{X} = -C = const \tag{5.11.13}$$

Let's consider the second equation together with the boundary conditions given by Eq. (5.11.10)

$$X'' + \lambda X = 0, \qquad X'(0) = 0, \qquad X'(L) = 0$$

where $\lambda = C/\alpha$. *This is the Sturm-Liouville problem*: If $\lambda = \nu^2 > 0$, then $X(x) = A\cos\nu x + B\sin\nu x$ and it can satisfy the b.c. if $B = 0$, $\nu L = n\pi$, $n = 0,1,2,\ldots$. The eigenfunctions $X_n(x)$, corresponding to eigenvalues λ_n, form an orthogonal system:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad X_n(x) = \cos\frac{n\pi x}{L}$$

$$C_n = \alpha \lambda_n = \alpha \left(\frac{n\pi}{L}\right)^2$$

Then a particular solution of the first Eq. in (5.11.13) for a given C_n takes the form:

$$\Theta'_n/\Theta_n = -C_n \qquad \Rightarrow \qquad \Theta_n(t) = \exp(-C_n t)$$

Now any $\Theta_n(t) X_n(x)$ is the solution of Eq. (5.11.11) that satisfies the boundary conditions (5.11.10). Due to linearity of the original equation (5.11.11) and boundary conditions, the solution can be represented in the form:

$$T(t,x) = \frac{T_0}{2} + \sum_{n=1}^{\infty} T_n \Theta_n(t) X_n(x) = \frac{T_0}{2} + \sum_{n=1}^{\infty} T_n e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t} \cos\frac{n\pi x}{L} \qquad (5.11.14)$$

Coefficients T_n should be found based on the initial condition given by Eq. (5.11.9):

$$T(0,x) = \tilde{T}_0(x) = \frac{T_0}{2} + \sum_{n=1}^{\infty} T_n \cos\frac{n\pi x}{L} \qquad (5.11.15)$$

Since the *RHS of Eq. (5.11.15) is the Fourier series* of the function $\tilde{T}_0(x)$, T_n can be determined as regular Fourier coefficients. Note that $\tilde{T}_0(x)$ is non-periodic, but can be naturally extended to an even periodic function with the period $2L$. It explains the absence of $\sin$-terms in (5.11.15).

Note: Depending on $p(x)$, $q(x)$, and $r(x)$, the eigenfunctions of the Sturm-Liouville problem can be not only trigonometric functions, but also many other functions. In particular, solutions of the boundary value problems for PDEs with separating variables in domains with axial symmetry result in eigenfunctions in the form of the **Legendre polynomials**.

5.12. Application of the Fourier series: Frequency spectrum analysis

Let's assume that we represented some P-periodic function in the form of the Fourier series:

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega x}, \qquad c_k = \frac{1}{P}\int_{a}^{a+P} f(x) e^{-ik\omega x} dx \tag{5.12.1}$$

if the independent variable is time, $x = t$, then we call $f(t)$ a **signal** in the **time domain**, and $\omega_k = k\omega$ is the **angular frequency**.

Spectrum (spectral) analysis, also referred to as **frequency domain analysis** or **spectral density estimation**, is the technical process of decomposing a complex signal into simpler parts. Any process that quantifies the various amounts (e.g. amplitudes, energies, powers, intensities, or phases), versus frequency can be called **spectrum analysis**.

The Fourier series gives us a signal as a combination of simpler parts. Every part has the form of a harmonic oscillation. We can look at c_k as at measure of importance of individual harmonic oscillation of given frequency ω_k in the signal.

Every complex amplitude c_k can be represented in the form including **amplitude** C_k and **phase** φ_k:

$$k > 0:\ c_k = \frac{a_k - ib_k}{2} = C_k e^{i\varphi_k}, \qquad C_k = \frac{\sqrt{a_k^2 + b_k^2}}{2}, \qquad \tan\varphi_k = -\frac{b_k}{a_k}$$

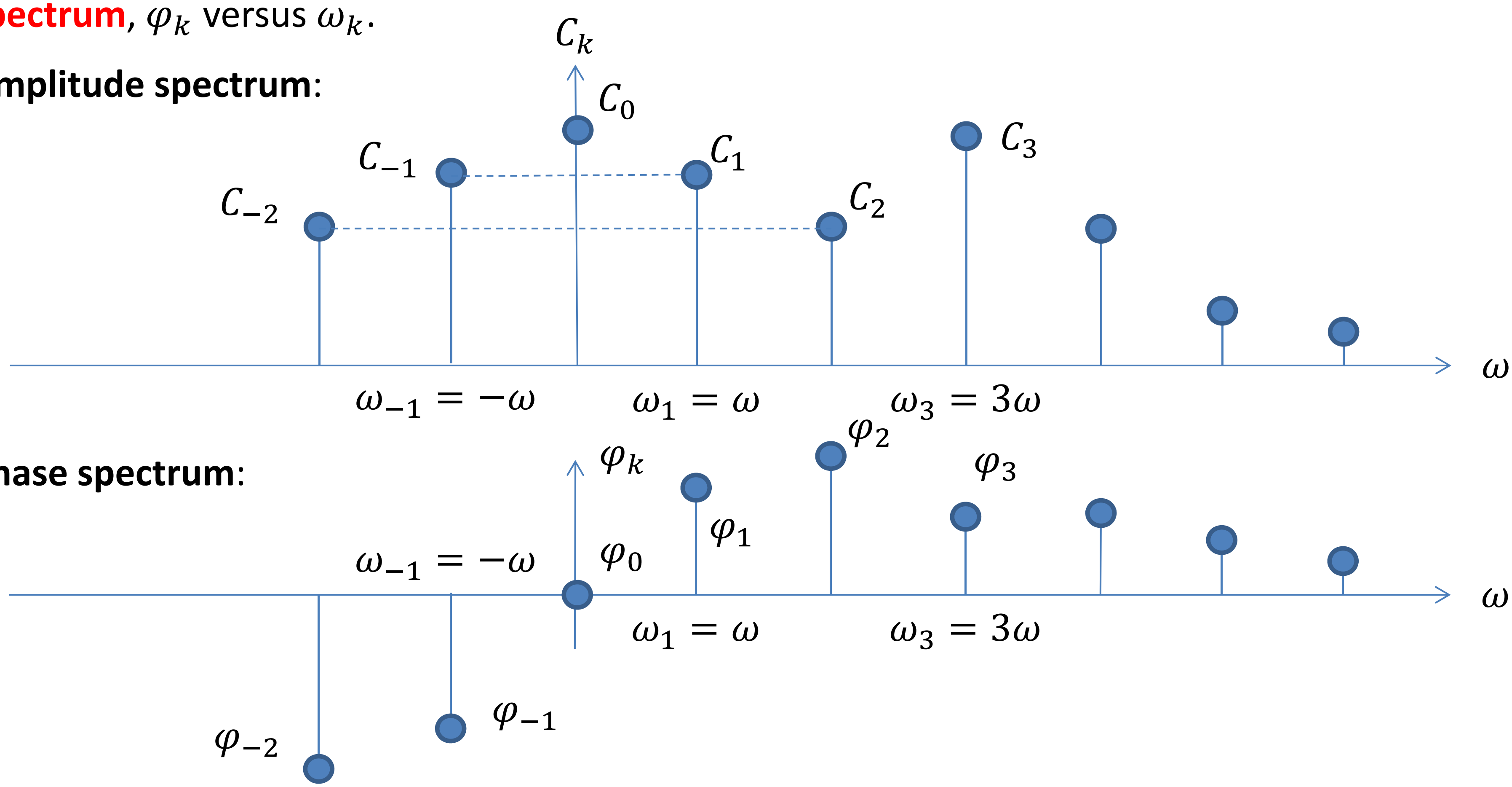
$$k < 0:\ c_k = \frac{a_{-k} + ib_{-k}}{2} = C_k e^{i\varphi_k}, \qquad C_k = \frac{\sqrt{a_{-k}^2 + b_{-k}^2}}{2}, \qquad \tan\varphi_k = \frac{b_{-k}}{a_{-k}} \tag{5.12.2}$$

5.12. Application of the Fourier series: Frequency spectrum analysis

Spectral analysis implies that we plot the **amplitude spectrum**, C_k versus ω_k, and **phase spectrum**, φ_k versus ω_k.

Amplitude spectrum:

Phase spectrum:



Note 1: Amplitude spectrum is the even function, phase spectrum is the odd function.

Note 2: Spectrum Fourier analysis of periodic functions results in discrete spectrums.

Note 3: Amplitude C_k can be thought as a measure of representativeness of oscillations with given frequency ω_k in the signal.

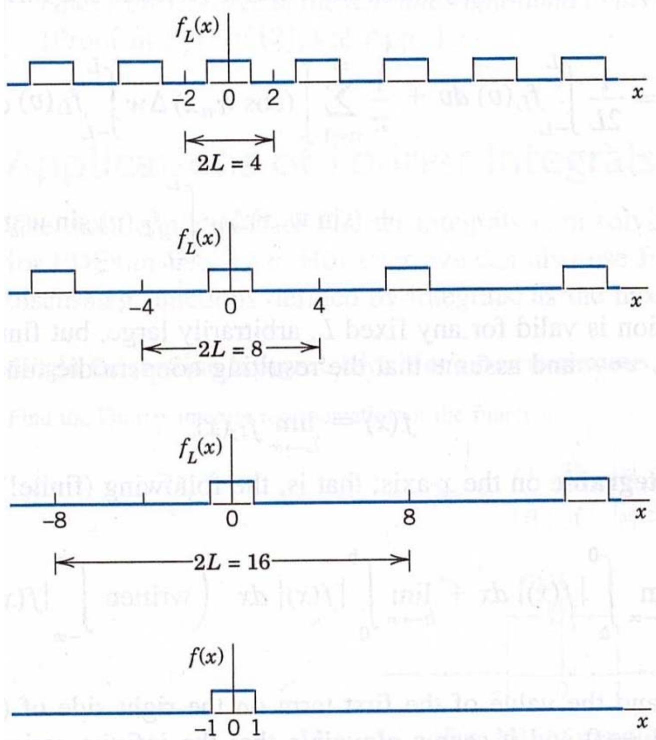
5.12. Application of the Fourier series: Frequency spectrum analysis

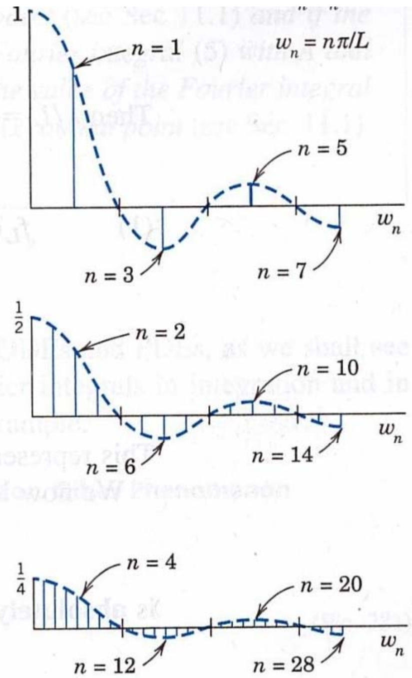
Example: Periodic rectangular wave of period $2L > 2$. What happens if $L \to \infty$?

Signal $f_L(x)$

Amplitude spectrum $a_n(\omega_n)$

$$\omega_n = \frac{\pi n}{L}$$





$$f_L(x) = \begin{cases} 0 & \text{if} \quad -L < x < -1 \\ 1 & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad 1 < x < L. \end{cases}$$

The function is even, $b_n = 0$, $C_0 = |a_0|/2, C_k = |a_k|, \varphi_k = 0$

We can analyze the amplitude spectrum in terms of a_n

$$a_0 = \frac{1}{L}\int_{-1}^{1} dx = \frac{2}{L}$$

$$a_n = \frac{1}{L}\int_{-1}^{1} \cos\frac{n\pi x}{L}\, dx = \frac{2}{L}\int_{0}^{1} \cos\frac{n\pi x}{L}\, dx = \frac{2}{L}\,\frac{\sin(n\pi/L)}{n\pi/L}.$$

$$f(x) = \lim_{L\to\infty} f_L(x) = \begin{cases} 1 & \text{if} -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Non-periodic function obtained at $L \to \infty$

Note 1: The difference between neighbor frequencies depends on the half-period L: $\Delta\omega = \omega_{n+1} - \omega_n = \pi/L$, a_n are proportional to $\Delta\omega$.

Note 2: In the limit of the non-periodic function, when $P \to \infty$, $\Delta\omega \to 0$ and the discrete spectrum evolves into the **continuous** one.

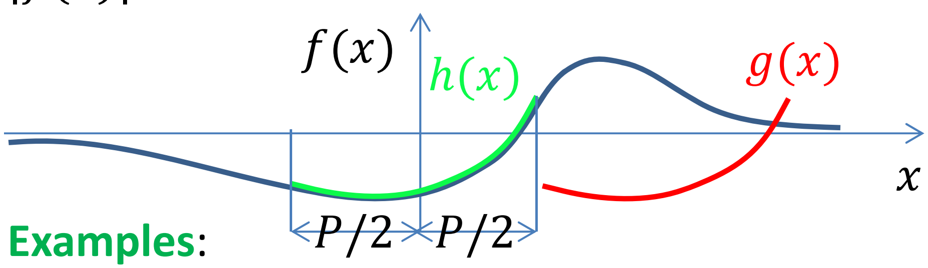
5.13. Fourier transform

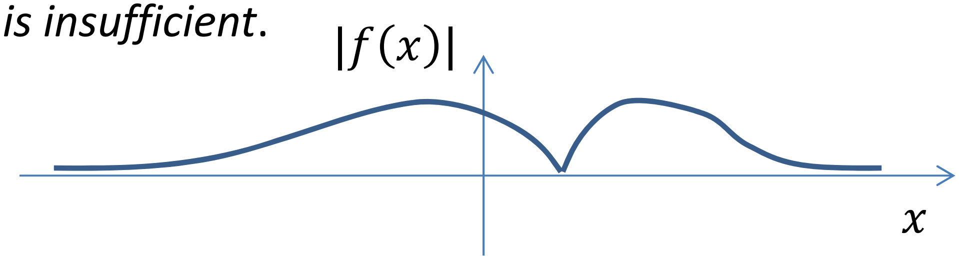
With the Fourier series we can study properties of periodic functions, or periodic extensions of non-periodic functions given at a finite interval. *The motivation of the Fourier transform is to extend the developed approach to non-periodic functions* and, in particular, to non-periodic functions $f(x)$ that can attain non-zero values at arbitrary x.

We limit our consideration by **absolutely integrable** functions $f(x)$, i.e. such functions for which the following integral exists

$$\int_{-\infty}^{\infty} |f(x)| dx < +\infty \qquad (5.13.1)$$

It means that the area between the plot of $|f(x)|$ and x-axis is finite. It is possible only if $|f(x)| \rightarrow 0$ when $x \rightarrow \infty$. *But this last condition is insufficient*.





Examples:

1. $f(x) = \exp(-a|x|)$ is absolutely integrable function.
2. $f(x) = 1/|x|$ is not absolutely integrable function.

In order to find equations of the Fourier transform, let's consider some absolutely integrable function $f(x)$, choose some arbitrary $P > 0$, and then

1. Introduce a new function given in a finite interval $h(x) = f(x)$ at $-P/2 \leq x \leq P/2$.
2. Introduce P-periodic extension $g(x)$ of $h(x)$.

5.13. Fourier transform

The periodic extension can be expanded into the Fourier series. According to the Dirichlet theorem, in the interval $-P/2 \le x \le P/2$, the Fourier series for $g(x)$ coincides with $f(x)$ with exception of points of discontinuities, so we can write

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikwx}, \qquad c_k = \frac{1}{P} \int_{-P/2}^{P/2} f(x) e^{-ikwx} dx, \qquad -\frac{P}{2} \le x \le \frac{P}{2} \qquad (5.13.2)$$

where we use the complex representation of the Fourier series and notation $w = 2\pi/P$.

Now let's see how Eq. (5.13.2) evolves when $P \to \infty$.

The complex amplitude c_k corresponds to the oscillation with angular frequency $\omega_k = kw = 2\pi k/P$. These frequencies form a discrete spectrum. The difference between neighbor frequencies $\Delta\omega = \omega_{k+1} - \omega_k = 2\pi/P \to 0$ when $P \to \infty$. Thus, the limit of non-periodic function $f(x)$ (obtained at $P \to \infty$) is characterized by a **continuous spectrum** when the angular frequency can attain any real value. Then let's re-write Eq. (5.13.2) as follows

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{i\omega_k x}, \qquad c_k = \frac{\Delta\omega}{2\pi} \int_{-P/2}^{P/2} f(x) e^{-i\omega_k x} dx$$

Note that $|c_k| \sim \Delta\omega \to 0$ when $P \to \infty$. The next step is to introduce $\hat{f}(\omega_k) = \sqrt{2\pi} c_k / \Delta\omega$:

$$\hat{f}(\omega_k) = \frac{1}{\sqrt{2\pi}} \int_{-P/2}^{P/2} f(x) e^{-i\omega_k x} dx \qquad (5.13.3)$$

5.13. Fourier transform

Then

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{f}(\omega_k) e^{i\omega_k x} \Delta\omega \qquad (5.13.4)$$

Note that so far Eqs. (5.13.3) and (5.13.4) are equivalent to Eq. (5.13.2), but they allow us to consider the limit $P \to \infty$. In this limit, the RHS of Eq. (5.13.4) becomes the Riemann integral sum, i.e. it transforms to the integral with the integrand $\hat{f}(\omega_k) e^{i\omega_k x}$. Thus, in the limit $P \to \infty$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \qquad (5.13.5)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \qquad (5.13.6)$$

If $\hat{f}(\omega)$ exists, then it is called the **Fourier (integral) transform** of $f(x)$, representation of $f(x)$ in the form of Eq. (5.13.6) is called the **inverse Fourier transform** or **Fourier integral** of $f(x)$.

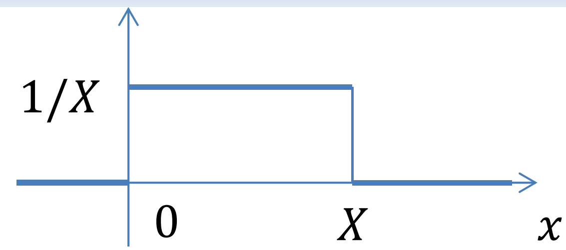
Note 1: The Fourier transform is the complex-valued function.

Note 2: *The Fourier transform can be formulated in many forms* that usually differ from each other by the choice of other variables instead of ω and different coefficients before the integrals instead of $1/\sqrt{2\pi}$. It can be also formulated in purely real form. The different forms of the Fourier transform will be considered in section 5.14.

5.13. Fourier transform

Example:

$$f(x) = \begin{cases} 1/X & 0 \le x \le X \\ 0 & x < 0, x > X \end{cases}$$



Let's calculate the Fourier transform:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{X} f(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi} X} \int_{0}^{X} e^{-i\omega x} dx$$

Let's use the Euler formula $e^{-i\omega x} = \cos \omega x - i \sin \omega x$:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi} X} \left(\int_{0}^{X} \cos \omega x \, dx - i \int_{0}^{X} \sin \omega x \, dx \right) = A(\omega) - iB(\omega)$$

where

$$A(\omega) = \frac{1}{\sqrt{2\pi} X} \int_{0}^{X} \cos \omega x \, dx = \frac{1}{\sqrt{2\pi} X \omega} \sin \omega x \Big|_{x=0}^{x=X} = \frac{\sin \omega X}{\sqrt{2\pi} X \omega}$$

$$B(\omega) = \frac{1}{\sqrt{2\pi} X} \int_{0}^{X} \sin \omega x \, dx = -\frac{1}{\sqrt{2\pi} X \omega} \cos \omega x \Big|_{x=0}^{x=X} = \frac{1 - \cos \omega X}{\sqrt{2\pi} X \omega} = \frac{2\sin^2(\omega X/2)}{\sqrt{2\pi} X \omega}$$

Note that $A(\omega)$ *is the even function and* $B(\omega)$ *is the odd function.*

5.13. Fourier transform

The conditions when $f(x)$ can be represented by the Fourier integral are given by the following:

Theorem: Fourier inverse theorem

Let's consider a function $f(x)$, which satisfies the following conditions:

1. $f(x)$ is absolutely integrable.
2. $f(x)$ is a piecewise continuous in every finite interval.
3. $f(x)$ has finite left-hand, $f'(x-0)$, and right-hand, $f'(x+0)$, derivatives in every point, i.e.

$$f'(x_0-0)=\lim_{h>0,h\to 0}\frac{f(x_0-0)-f(x_0-h)}{h},\ f'(x_0+0)=\lim_{h>0,h\to 0}\frac{f(x_0+h)-f(x_0+0)}{h}$$

Then

1. The Fourier transform exists

$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx$$

2. In every point x, where $f(x)$ is continuous, $f(x)$ can be represented by the Fourier integral,

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega$$

3. At a point x, where $f(x)$ is discontinuous,

$$\frac{1}{2}[f(x+0)+f(x-0)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega$$

5.13. Fourier transform

Note 1. Existence of left-hand, $f'(x-0)$, and right-hand, $f'(x+0)$, derivatives implies existence of left-hand, $f\,(x-0)$, and right-hand, $f\,(x+0)$, limits in every point. In other words, only functions with jump discontinuities are allowed in the Fourier inverse theorem.

Note 2. The conditions of the Fourier inverse theorem are similar to the Dirichlet conditions, but more restrictive. In the Fourier inverse theorem, it is required additionally

a. Existence of left- and right-hand side derivatives (only existence of left- and right-hand limits is requited by the Dirichlet conditions).
b. The function $f(x)$ should be absolutely integrable (Note that any non-zero periodic functions is not absolutely integrable).

5.14. Various forms of the Fourier transform

Fourier transform and Fourier integral

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx, \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \qquad (5.14.1)$$

are used in many different forms. A few of such forms are considered below. We will mark these different forms of the Fourier transform by the subscript "*" in order to distinguish them from our basic form given by Eq. (5.14.1).

1. Form based on the "true" frequency $\xi = \omega/(2\pi)$

Then $\omega = 2\pi\xi, d\omega = 2\pi d\xi$ and

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i2\pi\xi x} dx, \qquad f(x) = \frac{2\pi}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i2\pi\xi x} d\xi$$

Now let's introduce

$$\hat{f}_*(\xi) = \sqrt{2\pi} \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi\xi x} dx \qquad (5.14.2a)$$

Then

$$f(x) = \sqrt{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i2\pi\xi x} d\xi = \int_{-\infty}^{\infty} \hat{f}_*(\xi) e^{i2\pi\xi x} d\xi \qquad (5.14.2b)$$

5.14. Various forms of the Fourier transform

2. Non-symmetric form I

Let's introduce

$$\hat{f}_*(\xi) = \frac{\hat{f}(\omega)}{\sqrt{2\pi}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \qquad (5.14.3a)$$

Then

$$f(x) = \int_{-\infty}^{\infty} \hat{f}_*(\xi)e^{i\omega x}d\omega \qquad (5.14.3b)$$

3. Non-symmetric form II

Let's introduce

$$\hat{f}_*(\omega) = \sqrt{\frac{2}{\pi}}\hat{f}(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \qquad (5.14.4a)$$

Then

$$f(x) = \frac{1}{2}\int_{-\infty}^{\infty} \hat{f}_*(\omega)e^{i\omega x}d\omega \qquad (5.14.4b)$$

4. Two-component form. Real form of the Fourier integral

Let's use the Euler formula $e^{-i\omega x} = \cos\omega x - i\sin\omega x$. Then Eq. (5.14.4a) reduces to

$$\hat{f}_*(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)(\cos\omega x - i\sin\omega x)dx = A(\omega) - iB(\omega), \qquad (5.14.5a)$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx\,, \qquad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx \qquad (5.14.5b)$$

Note that $A(-\omega) = A(\omega)$, $B(-\omega) = -B(\omega)$, i.e. $A(\omega)$ and $B(\omega)$ are always even and odd functions, correspondingly. Now let's use the Euler formula in Eq. (5.14.4b):

$$f(x) = \frac{1}{2}\int_{-\infty}^{\infty}[A(\omega) - iB(\omega)](\cos\omega x + i\sin\omega x)d\omega =$$

$$\frac{1}{2}\int_{-\infty}^{\infty}[A(\omega)\cos\omega x - \cancel{iB(\omega)\cos\omega x} + \cancel{iA(\omega)\sin\omega x} + B(\omega)\sin\omega x]d\omega$$

Since both $B(\omega)\cos\omega x$ and $A(\omega)\sin\omega x$ are odd functions, finally we have

$$f(x) = \frac{1}{2}\int_{-\infty}^{\infty}[A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega \qquad (5.14.5c)$$

5.14. Various forms of the Fourier transform

But now we see that the integrand, $A(\omega)\cos\omega x + B(\omega)\sin\omega x$, is the even function, so we can rewrite the last equation as

$$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega \qquad (5.14.5d)$$

Note: *Eqs. (5.14.5b)-(5.14.5d) allows us to formulate the Fourier integral in the purely real form* (without complex numbers).

5. Fourier transform for even and odd functions

If $f(x)$ is even, then $B(\omega) = 0$ and $f(x)\cos\omega x$ is the even function, so we can write

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx\,, \qquad f(x) = \int_0^\infty A(\omega)\cos\omega x\,d\omega \qquad (5.14.6)$$

The Fourier integral in the form of Eq. (5.14.6) is called the **Fourier cosine integral**.

If $f(x)$ is odd, then $A(\omega) = 0$ and $f(x)\sin\omega x$ is the even function, so we can write

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\,dx\,, \qquad f(x) = \int_0^\infty B(\omega)\sin\omega x\,d\omega \qquad (5.14.7)$$

The Fourier integral in the form of Eq. (5.14.7) is called the **Fourier sine integral**.

6. Fourier transform for half-range functions given at $x > 0$

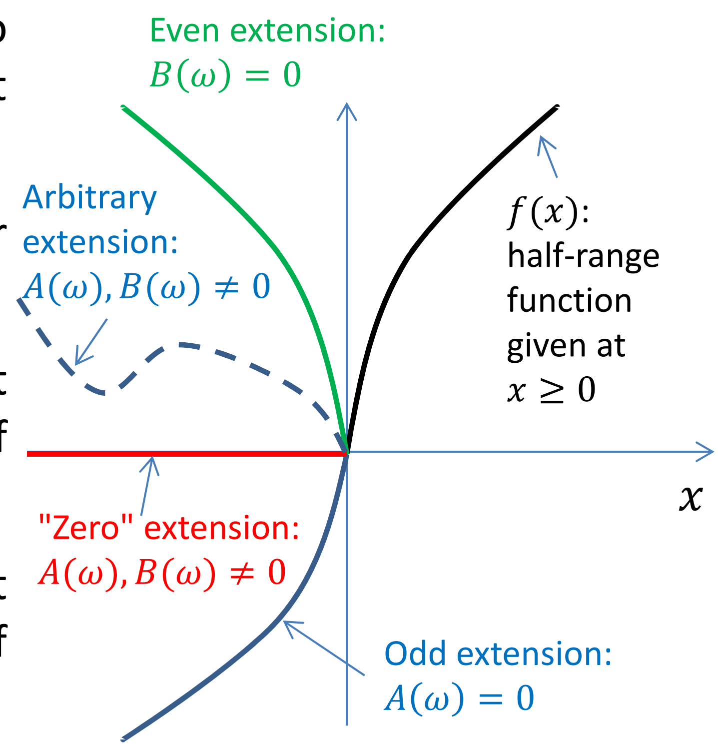
The case of functions given only in the half-range, e.g., only at $x > 0$, is an important case, since signals (functions of time, $x = t$) are recorded and studied starting at some initial time.

If values of $f(x)$ are given only at $x > 0$, then, in order to introduce its Fourier transform and integral, we have at least three options:

1. To assume that $f(x) = 0$ at $x < 0$. It gives us the Fourier integral in the general form of Eqs. (5.14.5b) and (5.14.5d).

2. To extend $f(x)$ for $x < 0$ evenly, i.e. to assume that $f(x) = f(-x)$ at $x < 0$. It gives us the representation of $f(x)$ in the form of the Fourier cosine integral, Eq. (5.14.6).

3. To extend $f(x)$ for $x < 0$ oddly, i.e. to assume that $f(x) = -f(-x)$ at $x < 0$. It gives us the representation of $f(x)$ in the form of the Fourier sine integral, Eq. (5.14.7).



Note: *Different extensions of $f(x)$ at $x < 0$ produce different Fourier transforms, but the Fourier integral in all these cases will have the same values at all $x > 0$* except points where $f(x)$ is discontinuous. It is guaranteed by the Fourier inverse theorem. In particular, the value of the Fourier integral at $x = 0$ can be different depending on whether the extended function has or does not have a discontinuity in this point.

5.15. Applications of the Fourier transform

Like the Fourier series, the Fourier transform is used in order to
1. Solve initial- and boundary value problems for ODEs and PDEs.
2. Perform spectrum (spectral) analysis.
3. Perform signal processing (Filtering, etc.).

Fourier transforms of many "basic" functions are tabulated like derivatives and antiderivatives (see Kreyszig, pages 534-536).

If the independent variable is the time, $x = t$, then $f(t)$ is often called the **signal** in the **time domain**. The Fourier transform $\hat{f}(\omega)$ is considered as the **image** of the signal in the **frequency domain**.

Spectrum analysis

The purpose of spectrum analysis is to decompose a given signal into simple, harmonic components, and to define, e.g., the dominant frequencies, i.e. frequencies of harmonic oscillations that provide major contributions to the signal.

The Fourier transform can be represented in the form (compare Eqs. (5.14.4a) and (5.14.5a))

$$\hat{f}(\omega) = \sqrt{\frac{\pi}{2}}[A(\omega) - iB(\omega)] = S(\omega)e^{i\varphi(\omega)}$$

where $S(\omega)$ and $\varphi(\omega)$ are the **amplitude** and **phase**

$$S(\omega) = \sqrt{\frac{\pi}{2}[A^2(\omega) + B^2(\omega)]}, \qquad \tan\varphi(\omega) = -\frac{B(\omega)}{A(\omega)}$$

5.15. Applications of the Fourier transform

Thus, the general complex Fourier transform corresponds to two real spectra: **Amplitude spectrum** $S(\omega)$ and **phase spectrum** $\varphi(\omega)$

If $f(x)$ is the even function, then $\hat{f}(\omega) = \sqrt{\pi/2}A(\omega)$ is a real-valued function and only the amplitude spectrum is of interest.

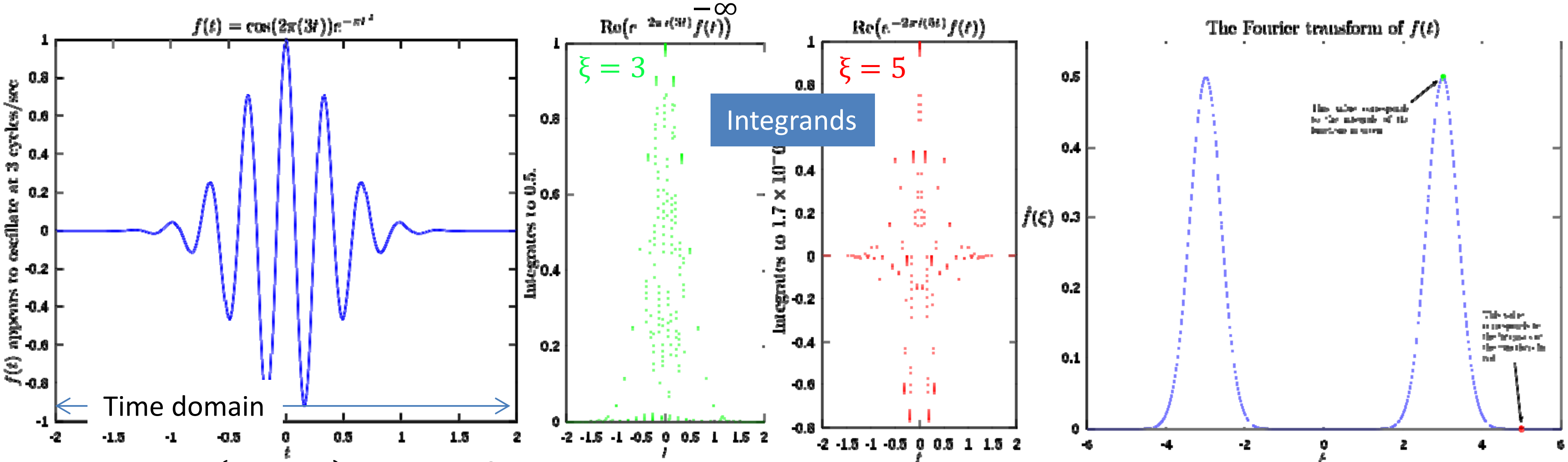
The amplitude $S(\omega)$ can be thought as a measure of representativeness of oscillations with given frequency ω in the signal.

Example: See http://en.wikipedia.org/wiki/Fourier_transform

Signal (oscillation of frequency $\xi = 3$ with varying magnitude)

$$\hat{f}_*(\xi) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi\xi t} dt$$

Amplitude spectrum (dominated by the oscillation of frequency $\xi = 3$)



$f(t) = \cos\big(2\pi(3t)\big)\exp(-\pi t^2)$ Dominant frequency Underrepresented frequency

Signal processing

The purpose of signal processing is to modify the input signal by changing the amplitudes, $S(\omega)$, or phases, $\varphi(\omega)$, of individual oscillation, i.e. by editing the signal in the frequency domain

Find the image (Fourier transform) → Edit the image in the frequency domain → Restore the corrected signal in the time domain (Fourier integral)

Practical goals:

1. Amplification of low-level signals in a given region of the spectrum
2. Noise reduction/cancelling (audio/video devices, analog radio, etc.)

Example: Noise reduction

5.16. Discrete Fourier transform (DFT). Fast Fourier transform (FFT)

In the majority of applications, the signal function $g(y)$ is given in the **discrete** (**tabulated**) form

y	y_0	y_1	y_2	…	y_{N-1}	y_{N-1}
$g(y)$	g_0	g_1	g_2	…	g_{N-2}	g_{N-1}

: N points

+ 1 point for periodicity

y_N
$g_N = g_0$

This is typical for signals obtained as results of measurements in physical experiments. We cannot perform signal processing or spectrum analysis of such signal by directly applying the Fourier transform, because calculation of the image

$$\hat{g}_*(\xi) = \int_{-\infty}^{\infty} g(x) e^{-i2\pi\xi x} dx$$

is possible only if we know $f(x)$ for all x. The Discrete Fourier transform (DFT) is a special modification of the Fourier transform that can be applied to discrete (tabulated) signals.

Reduction of a discrete signal to a standard form

Let's consider signals with evenly spaced points: $y_n = y_{n-1} + \Delta y$, $\Delta y = (y_{N-1} - y_0)/(N-1)$.
Let's turn the signal into the periodic function, assuming that there is an additional point $y_N = y_{N-1} + \Delta y$, where $g_N = g(y_N) = g_0$. Such extended signal $g(y)$ can be reduced to a "standard" periodic discrete signal $f(x)$ given in the interval $0 \le x \le 2\pi$:

$$x = 2\pi \frac{y - y_0}{y_N - y_0} \qquad f(x) = g\left(y_0 + \frac{y_N - y_0}{2\pi} x\right)$$

- For the standard discrete signal, $x_n = 2\pi n/N$ and $f_n = f(x_n) = g(y_n) = g_n$.
- **Standard discrete signal is given by the (column) vector f** $= [f_0, f_1, \dots, f_{N-1}]^T$.

Interpolation by trigonometric polynomials

Since $f(x)$ is given in a finite range $0 \le x \le 2\pi$, we could represent it in the form of complex Fourier series, Eq. (5.6.4), with $\omega = 2\pi/P = 1$:

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega x} = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

We cannot define infinitely large number of complex amplitudes c_k if we know values of $f(x)$ in only N points. Instead, let's approximate $f(x)$ with N-term complex trigonometric sum:

$$Q(x) = \sum_{k=0}^{N-1} c_k e^{ikx} = \sum_{k=0}^{N-1} c_k \left(e^{ix}\right)^k \tag{5.16.1}$$

Here we use $a^{xy} = (a^x)^y$

The function in the RHS of Eq. (5.16.1) is called the **complex trigonometric polynomial**, since it is an analog of "regular" polynomials in the form

$$P(x) = \sum_{k=0}^{N-1} c_k (x)^k$$

Our goal is to find c_k in Eq. (5.16.1) based on data in the vector $\mathbf{f} = [f_0, f_1, \ldots, f_{N-1}]^T$. We will do it, assuming that $Q(x)$ is the **interpolation polynomial**, i.e.

$$Q(x_n) = f(x_n) = f_n \tag{5.16.2}$$

Combining Eqs. (5.16.2) and (5.16.3), one can write that

$$\sum_{k=0}^{N-1} c_k \left(e^{i x_n}\right)^k = f_n \quad \Longrightarrow \quad \sum_{k=0}^{N-1} c_k \left(e^{i\frac{2\pi}{N}n}\right)^k = f_n \quad \Longrightarrow \quad \sum_{k=0}^{N-1} c_k \left(e^{i\frac{2\pi}{N}}\right)^{kn} = f_n$$

If we introduce $\overline{w} = e^{i\frac{2\pi}{N}}$, then the last equation can be re-written as follows:

$$\sum_{k=0}^{N-1} c_k \overline{w}^{kn} = f_n\,, \qquad n = 0, \dots, N-1 \qquad (5.16.3)$$

Eq. (5.16.3) is a linear system with respect to c_k and can be re-written in the matrix form as

$$\mathbf{Wc} = \mathbf{f}, \qquad \text{where} \qquad \mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{bmatrix}$$

and square matrix $\mathbf{W}$ has elements $W_{n,k} = \overline{w}^{kn}$. The solution this system is

$$\mathbf{c} = \mathbf{W}^{-1}\mathbf{f}$$

The elements $w_{n,k}$ of the inverse matrix $\mathbf{W}^{-1}$ are equal to (See Kreyszig, Sect. 11.9, p. 529)

$$w_{k,n} = \frac{1}{N} w^{kn}, \qquad \text{where} \qquad w = e^{-i\frac{2\pi}{N}}$$

(w and $\overline{w}$ are complex conjugate) and, thus

$$c_k = \sum_{n=0}^{N-1} w_{k,n} f_n = \frac{1}{N} \sum_{n=0}^{N-1} f_n \left(e^{-i\frac{2\pi}{N}}\right)^{kn} \quad \Longrightarrow \quad \boxed{c_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n e^{-ikx_n}} \qquad (5.16.4)$$

5.16. Discrete Fourier transform (DFT). Fast Fourier transform (FFT)

Eq. (5.16.1) and (5.16.4) defines the trigonometric interpolation polynomial which is used instead of trigonometric (Fourier) series for tabulated functions.

Discrete Fourier Transform (DFT)

We connected the Fourier transform $\hat{f}(\omega_k)$ with c_k as (See slide 46): $\hat{f}(\omega_k) = \sqrt{2\pi} c_k / \Delta\omega$. In the case of a tabulated function $\Delta\omega = \omega/N = 1/N$ and it is reasonable to introduce the **discrete Fourier transform (DFT)** of $\mathbf{f}$ as a (column) vector $\hat{\mathbf{f}} = [\hat{f}_0, \hat{f}_1, \dots, \hat{f}_{N-1}]^T$, where

$$\hat{f}_n = N c_k = \sum_{n=0}^{N-1} f_n e^{-ikx_n} \qquad (5.16.5)$$

The DFT can be also introduced in the matrix form

$$\hat{\mathbf{f}} = \mathbf{F}_N \mathbf{f} \qquad : \text{DFT} \qquad (5.16.6)$$

where the Fourier matrix $\mathbf{F}_N = N\mathbf{W}^{-1} = [e_{n,k}]$ has elements $e_{n,k} = w^{kn} = \left(e^{-i\frac{2\pi}{N}}\right)^{nk}$.

If we know the image of the signal, $\hat{\mathbf{f}}$, and want to find the signal itself, we just need to resolve the linear system given by Eq. (5.16.6) with respect to $\mathbf{f}$:

$$\mathbf{f} = \mathbf{F}_N^{-1} \hat{\mathbf{f}} \qquad : \text{Inverse DFT} \qquad (5.16.7)$$

where $\mathbf{F}_N^{-1} = (1/N)\,\mathbf{W} = [\bar{e}_{n,k}/N]$ or (compare with Eq. (5.16.4)):

$$f_n = \frac{1}{N} \sum_{k=0}^{N-1} \hat{f}_k e^{-ikx_n} \qquad (5.16.8)$$

5.16. Discrete Fourier transform (DFT). Fast Fourier transform (FFT)

- Fourier transform of discrete (tabulated) data $\mathbf{f}$ is a vector (table) $\hat{\mathbf{f}}$, not function.
- In applications, the number of points in the signal is usually large, $N \gg 1$. In this case the direct application of Eq. (5.16.6) is inefficient, since it requires $\sim(N-1) \times N \sim N^2$ arithmetic operations. Direct calculations of DFT at large N is extremely lengthy operation!

Fast Fourier Transform (FFT)

- **Fast Fourier Transform (FFT)** is a DFT at **specific values** of N, where calculations are organized in a special manner that allows one to reduce the number of arithmetic operations.
- Usually in FFT, $N = 2^n$. In this case the number of arithmetic operations can be reduced to $N \log N$, e.g., $N = 1000$, $N^2 = 1000000$, $N \log N \approx 6900$,more than 100-fold acceleration!
- The most popular algorithm of FFT is the **Cooley-Tukey algorithm**. This method (and the general idea of an FFT) was popularized by a publication of J. W. Cooley and J. W. Tukey in 1965, but it was later discovered that those two authors had independently re-invented an algorithm known to Carl Friedrich Gauss around 1805. He developed an FFT-type algorithm to interpolate the orbits of asteroids Pallas and Juno from sample observations.
- See details in

  Kreyszig, Sect. 11.9, pp. 528-532

  https://en.wikipedia.org/wiki/Fast_Fourier_transform

  https://en.wikipedia.org/wiki/Cooley%E2%80%93Tukey_FFT_algorithm
- The majority of mathematical software has build-in capabilities for FFT. In MATLAB, FFT of tabulated data **X** can be performed with function **fft** (**X**). See

  https://www.mathworks.com/help/matlab/ref/fft.html?requestedDomain=www.mathworks.com