Chapter 6
Vector Calculus

Reading:
Kreyszig, *Advanced Engineering Mathematics, 10th Ed.,* 2011
Selection from chapters 9 and 10

Prerequisites:
Kreyszig, *Advanced Engineering Mathematics, 10th Ed.,* 2011

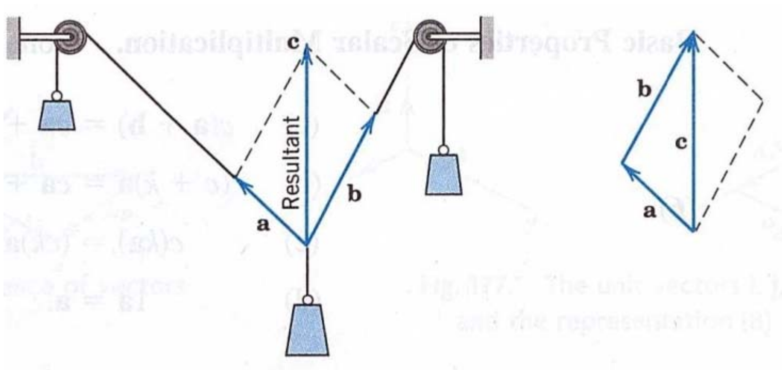
- Vector quantities. Dot and cross products: Sections 9.1-9.3
- Double integral: Section 10.3
- Triple integral: Section 10.7

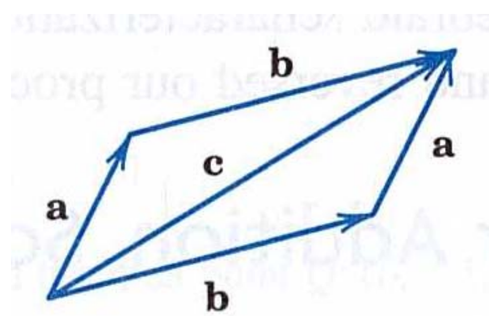
Contents

Topics for self-studying: Sections 6.3, 6.10, 6.11

6.1. Vector physical quantities

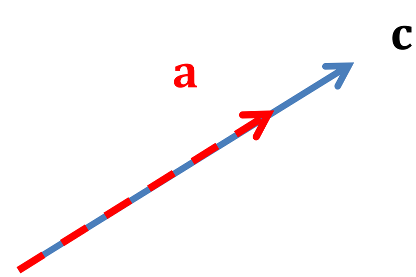
- Notion of a vector is inspired by the existence of physical quantities that are characterized by both magnitude and direction, e.g., velocity and force. In physics
  - ✓ The **scalar** a is a quantity that is determined by its magnitude (temperature).
  - ✓ The **vector** $\mathbf{a}$ is a quantity that is represented by a **directed line segment**, and thus, has **magnitude** $|\mathbf{a}|$ (**absolute value**, **length**, or **norm** of the vector) and direction (velocity, Force, angular momentum).
- Two vectors $\mathbf{a}$ and $\mathbf{b}$ are **equal** to each other, $\mathbf{a} = \mathbf{b}$, if they have the same length and direction even if their initial and terminal points are different. We say that any vector is a **free vector** if translation (displacement without rotation) does not change the vector.
- Two basic operations on vector physical quantities are the **vector addition** $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and **scalar multiplication** $\mathbf{c} = b\mathbf{a}$. Since physical vectors are defined geometrically, we introduce such operations also geometrically as





$\mathbf{c} = \mathbf{a} + \mathbf{b}$

Vector addition



$\mathbf{c} = b\mathbf{a} \; : \; |\mathbf{c}| = |b||\mathbf{a}|$

Scalar multiplication

6.1. Vector physical quantities

And assume that they posses the following properties

$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}, \qquad (\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c}), \qquad \mathbf{a}+\mathbf{0}=\mathbf{0}+\mathbf{a}=\mathbf{a}, \qquad a+(-\mathbf{a})=0$$
$$c(\mathbf{a}+\mathbf{b})=c\mathbf{a}+c\mathbf{b}, \qquad (a+b)\mathbf{c}=a\mathbf{c}+b\mathbf{c}, \qquad a(b\mathbf{c})=(ab)\mathbf{c}, \qquad 1\mathbf{a}=\mathbf{a} \tag{6.1.1}$$

Here:

- $\mathbf{0}$ is the **zero vector**, $|\mathbf{0}|=0$.
- $-\mathbf{a}$ has the same length as $\mathbf{a}$, but opposite direction.
- Existence of $-\mathbf{a}$ allows one to determine the **vector subtraction**: $\mathbf{c}=\mathbf{b}-\mathbf{a}\equiv\mathbf{b}+(-\mathbf{a})$.
- The **unit vector** is a vector of unit length, $|\mathbf{a}|=1$.

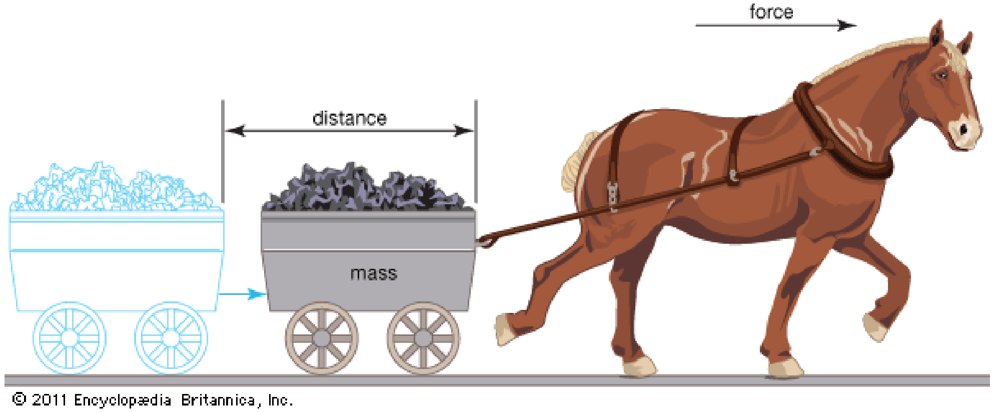
A whole set of objects (vectors) on which we can perform vector addition and scalar multiplication with properties given by Eqs. (6.1.1) is called the **(linear) vector space**.

- The idea behind the **vector calculus** is to utilize vectors and their functions for analytical calculations, i.e. calculations without geometrical considerations.
- It is possible if any vector is completely represented it terms of numbers, not directed line segments.

6.2. Vector calculus: Motivation and applications

Motivation: Calculation of work-type quantities

Work = force x distance

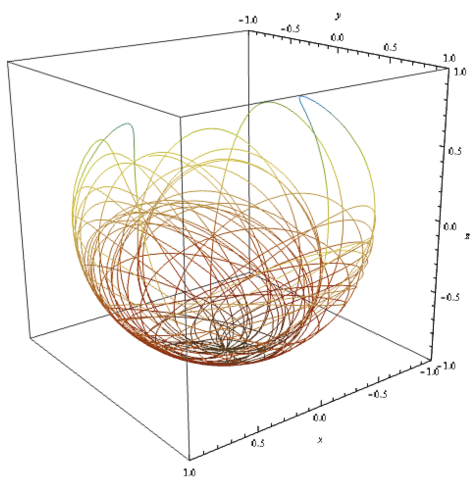


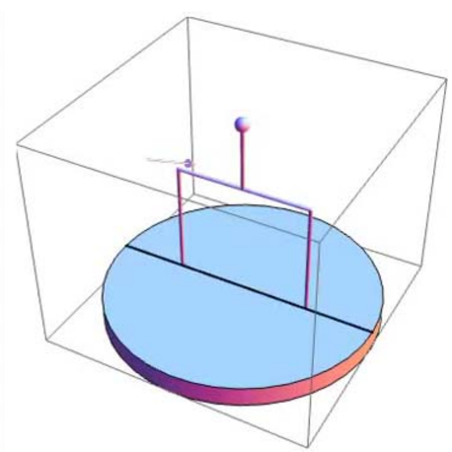
For general description of the work-type quantities we need:

- Mathematical description of curves
- Line integrals

What if the path is complex and/or force is not aligned with the velocity?

Trajectory of a chaotic pendulum





Line integrals are the ultimate generalization of equations

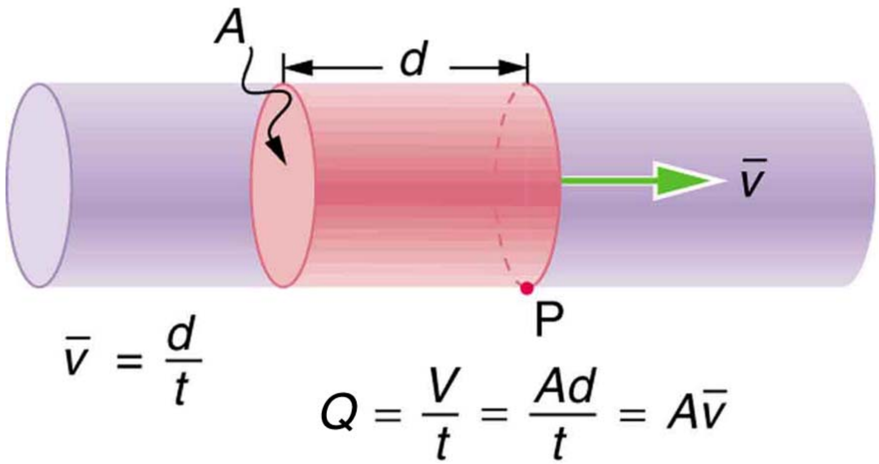
Work = force x distance

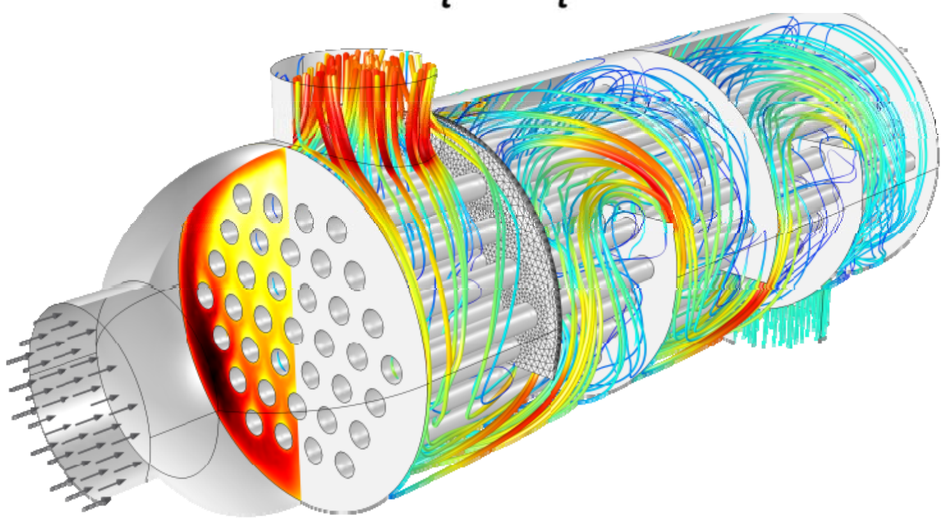
for work-type quantities

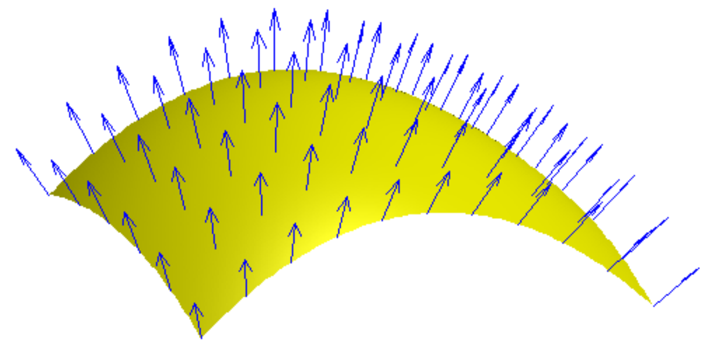
6.2. Vector calculus: Motivation and applications

Motivation: Calculation of flux-type quantities

Flux of a physical quantity is the amount of this quantity transferred through a given area (surface) per unit time







Surface integral is the ultimate generalization of equation

$$Q = A\,\overline{v}$$

for flux-type quantities

For general description of the flux-type quantities we need:

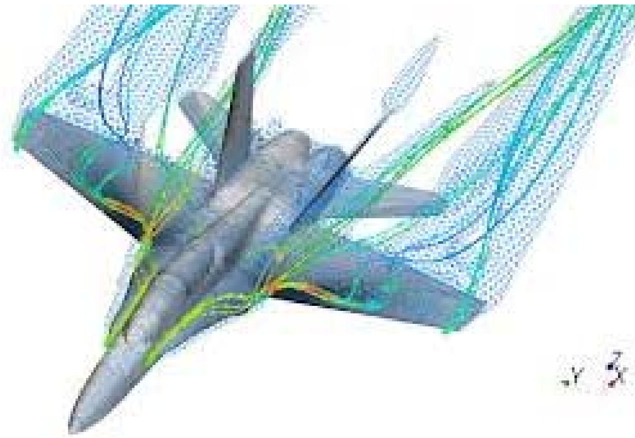
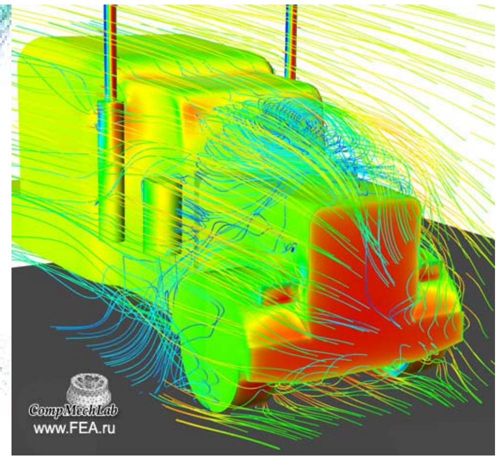
- Mathematical description of surfaces
- Surface integrals
- Relationships between flux- and work-type quantities (integral theorems)

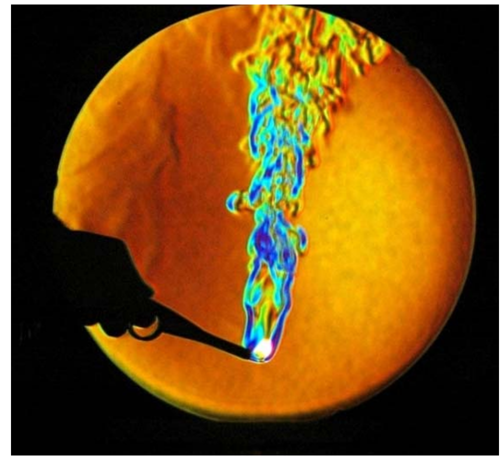
6.2. Vector calculus: Motivation and applications

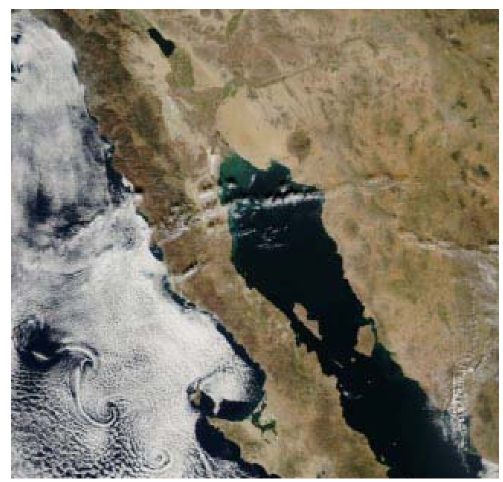
Applications of the vector calculus: All science and engineering fields where problems are formulated in terms of PDEs or require analysis of vector fields in multidimensional spaces.

- Formulation of physical laws in terms of scalar, vector, and tensor fields.
- General mathematical properties of such mathematical models .

Fluid mechanics and gas dynamics, combustion

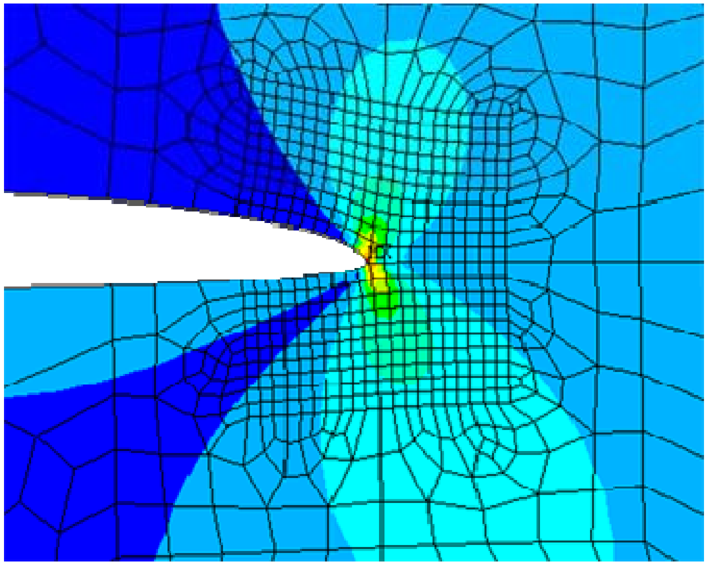



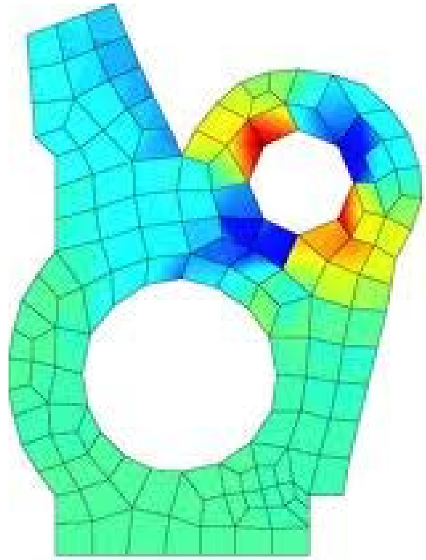


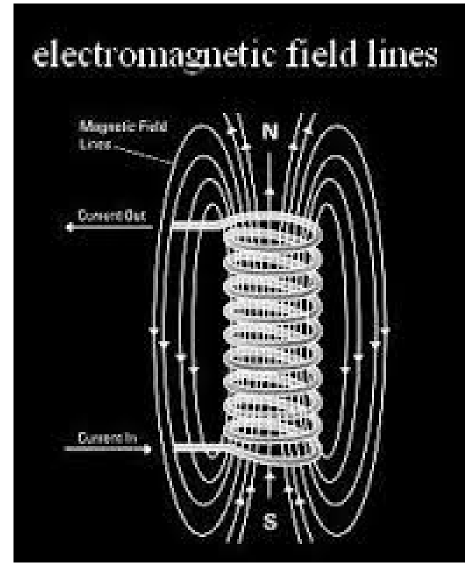
6.2. Vector calculus: Motivation and applications

Solid mechanics

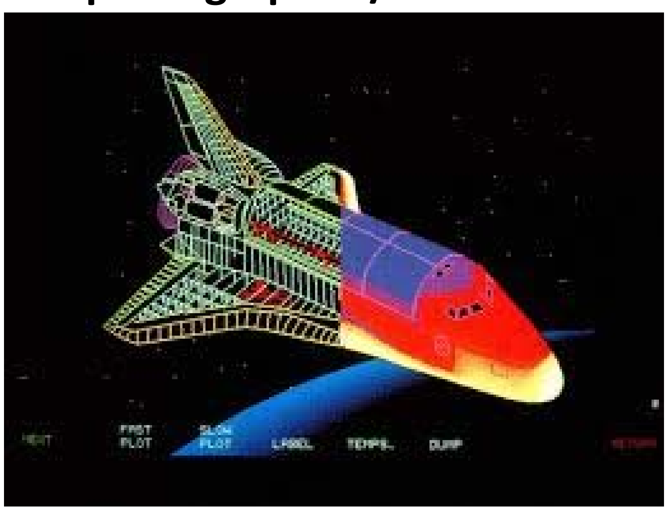




Electromagnetic theory

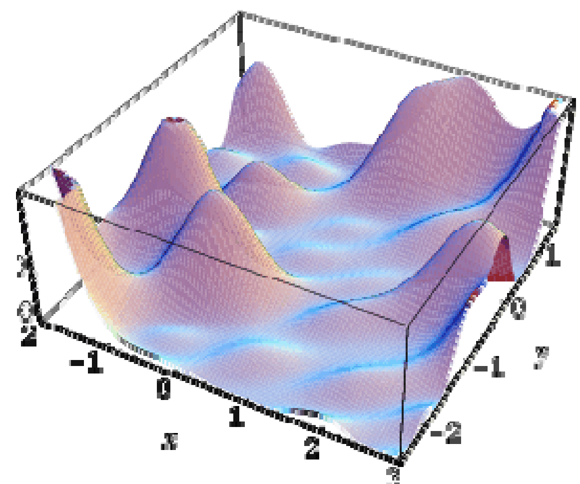


Computer graphics/CAD



Mapmaking

Optimization



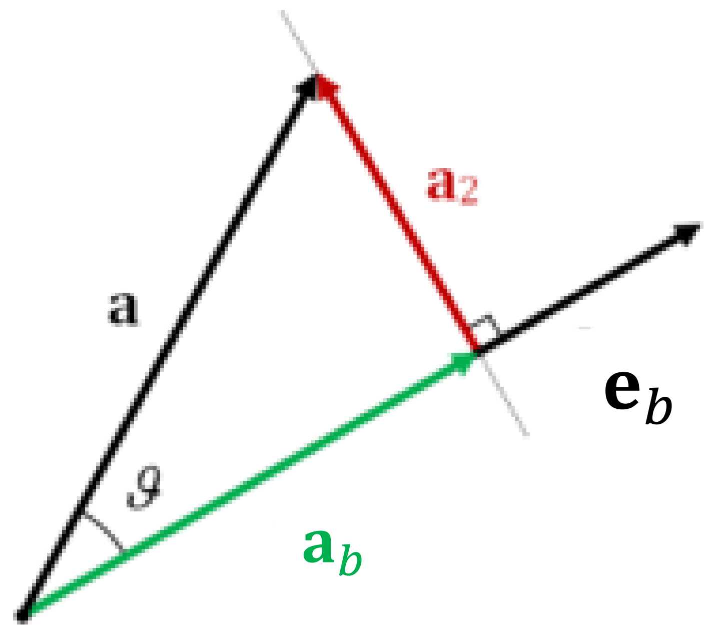
Statistical physics, Quantum mechanics, Rarefied gas dynamics, ...

6.3. Linear vector space. Dot, cross, and triple products

The idea behind *using the vector quantities in calculus* is that any vector can be represented by a few numbers that are called **components** of the vector. It allows us to perform all operation on vectors algebraically, i.e. without any geometrical considerations. Let's consider how we can introduce components of vectors.

First, let's introduce the geometrical projection of a vector onto an axis and postulate the existence of global 3D Cartesian coordinates

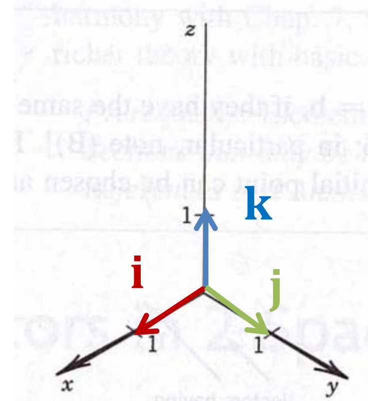
Let's consider a vector $\mathbf{a}$ and some **axis** (line with given direction). The direction of the axis can be characterized by a unit vector $\mathbf{e}_b$ placed along the line and having the same direction. Then we can geometrically define the **vector projection** $\mathbf{a}_b$ and **scalar projection** a_b



(6.3.1) $$a_b = |a| \cos\theta\,, \qquad \mathbf{a}_b = a_b \mathbf{e}_b$$

6.3. Linear vector space. Dot, cross, and triple products

Based on observations and our experience we can postulate that the physical space in the classical mechanics is **Euclidean**, i.e. we can introduce the **Cartesian coordinate system**, which is composed of three mutually perpendicular axis Ox, Oy, and Oz. Next we can fix an arbitrary linear scale and introduce three unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, directed along every axis, which we call the (Cartesian) **basis**.

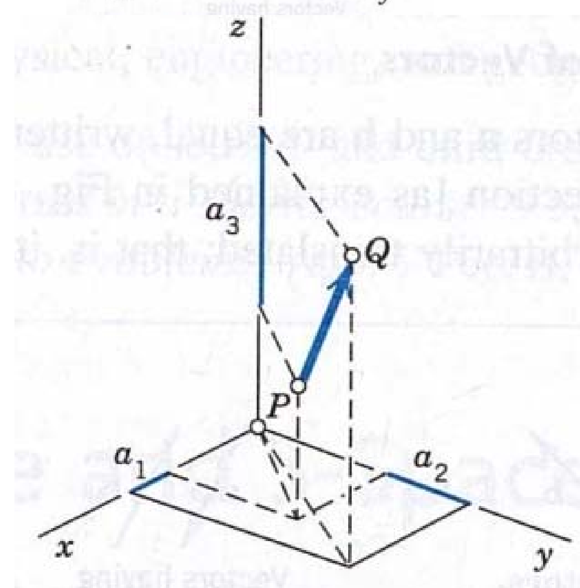


Then every vector can be uniquely represented in the form of a sum of its projections onto vectors of the Cartesian basis

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

where a_x, a_z, and a_z are called **Cartesian components** of vector $\mathbf{a}$. If the Cartesian coordinates are fixed, every vector (element of the vector space) is just a unique **ordered triple** of real numbers

(6.3.2) $$\mathbf{a} = a_x\mathbf{i} + a_z\mathbf{j} + a_z\mathbf{k} = (a_x, a_z, a_z)$$



Components (a_x, a_z, a_z) of vector $\mathbf{a}$ are determined by the basis $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and *if Cartesian coordinates change, the components or every vector change as well.*

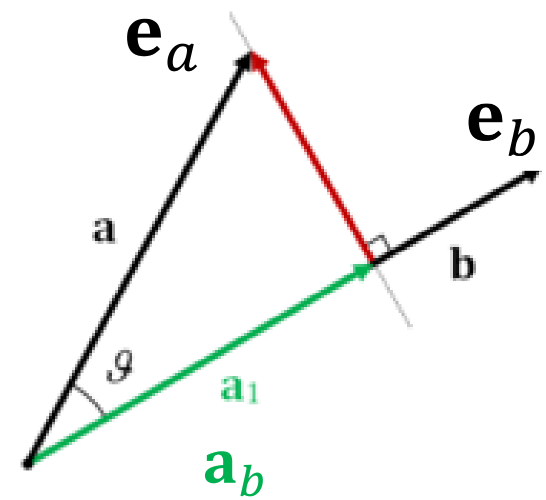
6.3. Linear vector space. Dot, cross, and triple products

Based on the Pythagorean theorem we see that the **norm** or **length** of the vector is now defined as

$$|\mathbf{a}| = \sqrt{a_x^2 + a_z^2 + a_z^2}$$

A vector $\mathbf{a}$ is called the **unit vector** if $|\mathbf{a}| = 1$.

The cosine of **angle** θ between vectors $\mathbf{a}$ and $\mathbf{b}$ ($0 \leq \theta \leq \pi$) is equal to the scalar projection e_{ab} of the unit vector $\mathbf{e}_a$ along $\mathbf{a}$ onto the unit vector $\mathbf{e}_b$ along $\mathbf{b}$.



The **dot (inner, scalar) product** $\mathbf{a} \cdot \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the number (scalar) defined as

(6.3.3) $$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta \,.$$

Obviously, $\mathbf{i} \cdot \mathbf{i} = 1$, $\mathbf{i} \cdot \mathbf{j} = 0$, etc., so that

(6.3.4) $$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$$

and

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{\mathbf{a}^2}$$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are called **orthogonal** if $\mathbf{a} \cdot \mathbf{b} = 0$.

Properties of the dot product:

1. $\mathbf{a} \cdot \mathbf{a} \geq 0$,

   if $\mathbf{a} \cdot \mathbf{a} = 0$ then $\mathbf{a} = \mathbf{0}$
1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
2. $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$
3. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b})$

6.3. Linear vector space. Dot, cross, and triple products

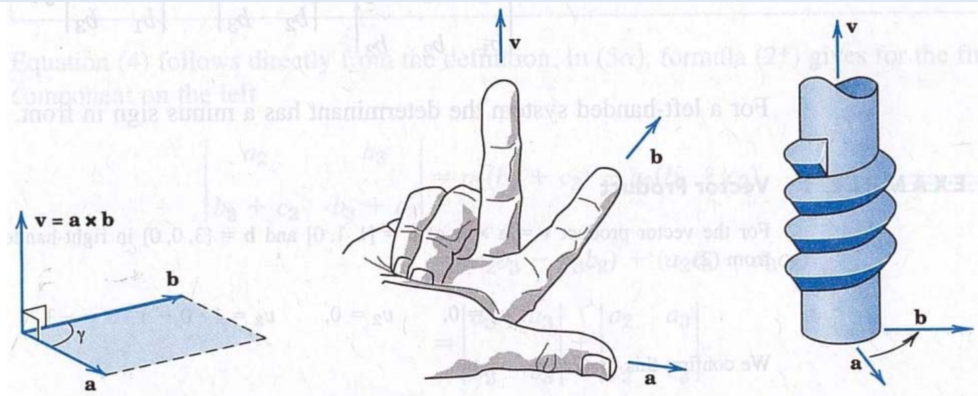
The **cross (vector) product** of vectors $\mathbf{a}$ and $\mathbf{b}$ is the vector $\mathbf{a} \times \mathbf{b}$ that

1. Is equal to zero vector $\mathbf{0}$ if $|\mathbf{a}| = 0$ or $|\mathbf{b}| = 0$.
2. Is orthogonal to $\mathbf{a}$ and $\mathbf{b}$.
3. Has the length

$$|\mathbf{a} \times \mathbf{b}| = |a||b||\sin\theta|$$

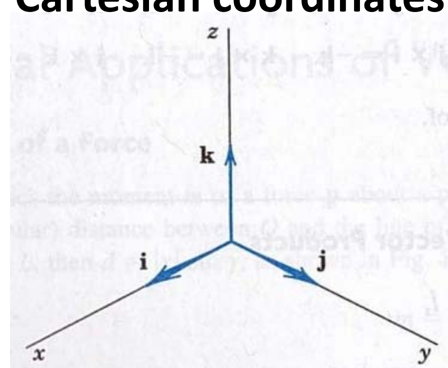
   where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

4. Has such direction that $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ is the **right-handed triplet**, i.e. rotation from $\mathbf{a}$ to $\mathbf{b}$ from the top of $\mathbf{v}$ occurs in the counter-clockwise direction.

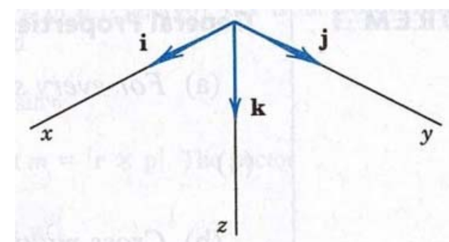


$|\mathbf{a} \times \mathbf{b}| = |a||b||\sin\theta|$= area of parallelogram

Right-handed Cartesian coordinates



Left-handed Cartesian coordinates



Then

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{a} \times \mathbf{b} = (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) \times (b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) = a_x b_x \, \mathbf{i} \times \mathbf{i} + a_x b_y \, \mathbf{i} \times \mathbf{j} + \ldots = \cancel{a_x b_x \mathbf{0}} + a_x b_y \, \mathbf{k} + \ldots$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_x & b_z \end{vmatrix} \qquad (6.3.10)$$

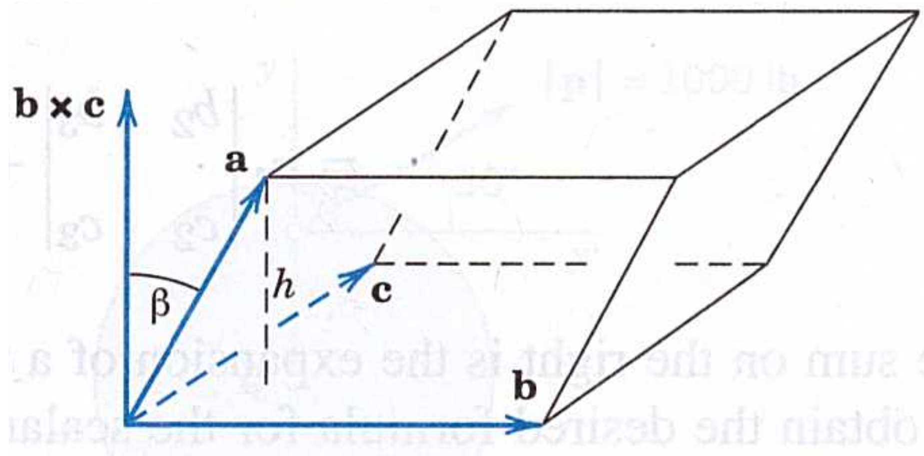
6.3. Linear vector space. Dot, cross, and triple products

Note: All properties of the vector product can be derived from the definition. It is important that

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}, \qquad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

The **triple product** $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ of vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the scalar calculated as

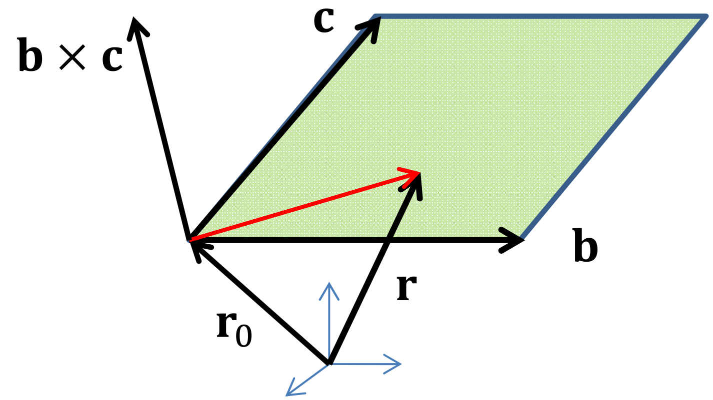
$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \qquad (6.3.11)$$



- The physical meaning of the triple product: The triple product is equal to the signed volume of a parallelepiped with edges along vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$:

  $$|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |\mathbf{a}||\cos\beta||\mathbf{b} \times \mathbf{c}| = h|\mathbf{b} \times \mathbf{c}|$$

- All other properties of the triple product can be easily derived from properties of dot and cross products.



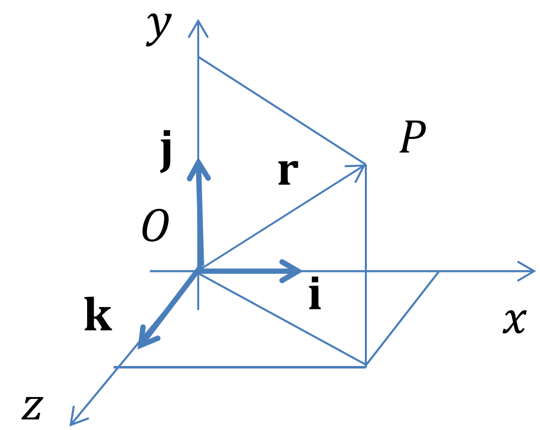
Example: Let's consider a plane given by two non-parallel vectors $\mathbf{b}$ and $\mathbf{c}$ lying in this plane and a point with the position vector $\mathbf{r}_0$. Then the equation of this plane is $(\mathbf{r} - \mathbf{r}_0) \cdot (\mathbf{b} \times \mathbf{c}) = 0$.

6.4. Scalar and vector fields and their derivatives

Let's consider 3D physical space. Based on experimental observations, in the classical mechanics we make the following assumption about properties of the physical space:

The physical space is a **Euclidean three-dimensional (3D) space** that means we can introduce (global, unique for the whole space) 3D **Cartesian coordinates** $Oxyz$ such that the distance between any two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by

$$l_{12} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



For every point P we can introduce the **position vector** $\mathbf{r} = \mathbf{r}(P)$ starting from the origin of the coordinates O and ending at point P.

We can introduce three mutually orthogonal **unit basic vectors** $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ connecting the origin O with points (1,0,0), (0,1,0) and (0,0,1) at the axes.
The the position vector can be represented in the form

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where (x, y, z) are coordinates and the **scalar (dot) product** is

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = x_1 x_2 + y_1 y_2 + z_2 z_1$$

Relative position of point P_2 with respect to point P_1 is characterized by the **relative position vector** $\Delta\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ with the length $l_{12} = \sqrt{\Delta\mathbf{r}_{12} \cdot \Delta\mathbf{r}_{12}}$.

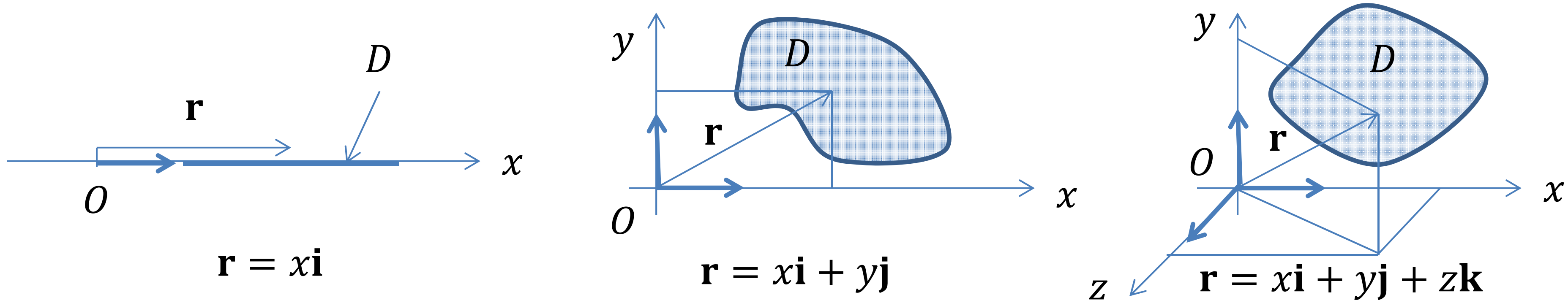
6.4. Scalar and vector fields and their derivatives

Let's consider some domain D, which can be

1. One-dimensional (1D): set of points at the real x-axis
2. Two-dimensional (2D): Set of points in xy-plane
3. Three-dimensional (3D): Set of points in xyz-space.

- We assume that we introduce a Cartesian coordinates $Oxyz$ with the global basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$
- In every case we can characterize a point P of D with the **position (radius) vector** $\mathbf{r}$
- In 3D, we will consider only right-hand coordinates

1D **2D** **3D**



$\mathbf{r} = x\mathbf{i}$ $\quad$ $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ $\quad$ $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Assume that for every point P from D we define a number

$$f = f(P) = f(\mathbf{r}) \qquad (6.4.1)$$

The function $f(\mathbf{r})$ is called the **scalar field**.

1D: $f = f(P) = f(\mathbf{r}) = f(x)$ is a function
2D: $f = f(P) = f(\mathbf{r}) = f(x, y)$ is a 2D (planar) scalar field
3D: $f = f(P) = f(\mathbf{r}) = f(x, y, z)$ is a 3D scalar field

6.4. Scalar and vector fields and their derivatives

The surface $f(x, y, z) = const$ is called the **isosurface** of the scalar field f.

In 2D case, curve defined by the equation $f(x, y) = const$ is called the **isoline** or **contour line**.

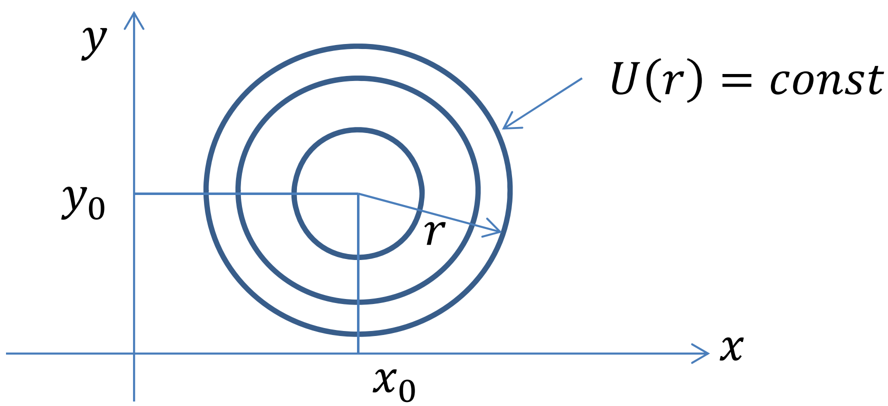
The isosurfaces and isolines are used to visualize scalar fields.

Example: Spherically symmetric gravitational field

The potential energy of a point mass m in the spherically symmetric field of mass M with center in point (x_0, y_0, z_0) is equal to

$$U(x, y, z) = -\frac{GMm}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} = -\frac{GMm}{r} = U(r)$$

Isosurfaces are spheres with center at (x_0, y_0, z_0).
Contour lines at xy-plane are circles with centers at (x_0, y_0).



6.4. Scalar and vector fields and their derivatives

Now let's assume that in every point P of D we define a vector $\mathbf{F}(P) = \mathbf{F}(\mathbf{r})$. Then $\mathbf{F}(\mathbf{r})$ is called the **vector field**:

$$\mathbf{F}(\mathbf{r}) = F_x(\mathbf{r})\mathbf{i} + F_y(\mathbf{r})\mathbf{j} + F_z(\mathbf{r})\mathbf{k} \qquad (6.4.2)$$

1D: $\mathbf{F} = \mathbf{F}(P) = \mathbf{F}(\mathbf{r}) = \mathbf{F}(x) = F_x(x)\mathbf{i}$
2D: $\mathbf{F} = \mathbf{F}(P) = \mathbf{F}(\mathbf{r}) = \mathbf{F}(x, y) = F_x(x, y)\mathbf{i} + F_y(x, y)\mathbf{j}$
3D: $\mathbf{F} = \mathbf{F}(P) = \mathbf{F}(\mathbf{r}) = \mathbf{F}(x, y, z) = F_x(x, y, z)\mathbf{i} + F_y(x, y, z)\mathbf{j} + F_z(x, y, z)\mathbf{k}$

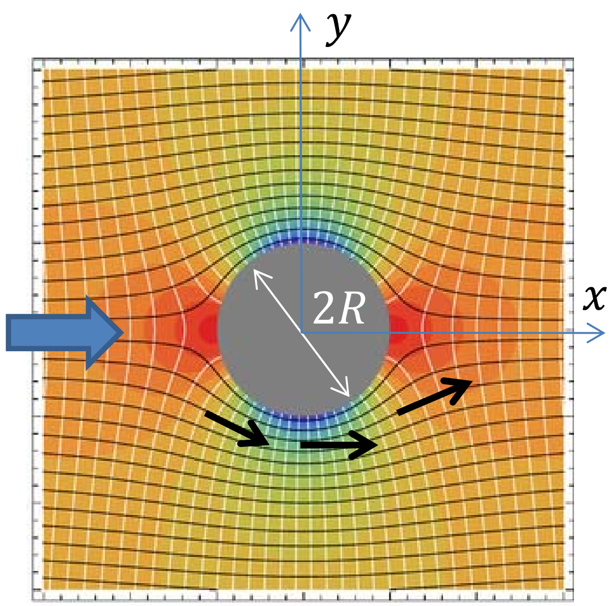
The curve at every point of which the vector field $\mathbf{F}$ is tangent to the curve is called the **field line** of the vector field $\mathbf{F}$. Field lines are used for visualization of vector fields.

Example 1: Potential flow field around a cylinder in a cross-flow.
r, θ are polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$)
Scalar field: Velocity potential

$$\varphi(r, \theta) = U\left(r + \frac{R^2}{r}\right)\cos\theta$$

Vector field: Fluid velocity

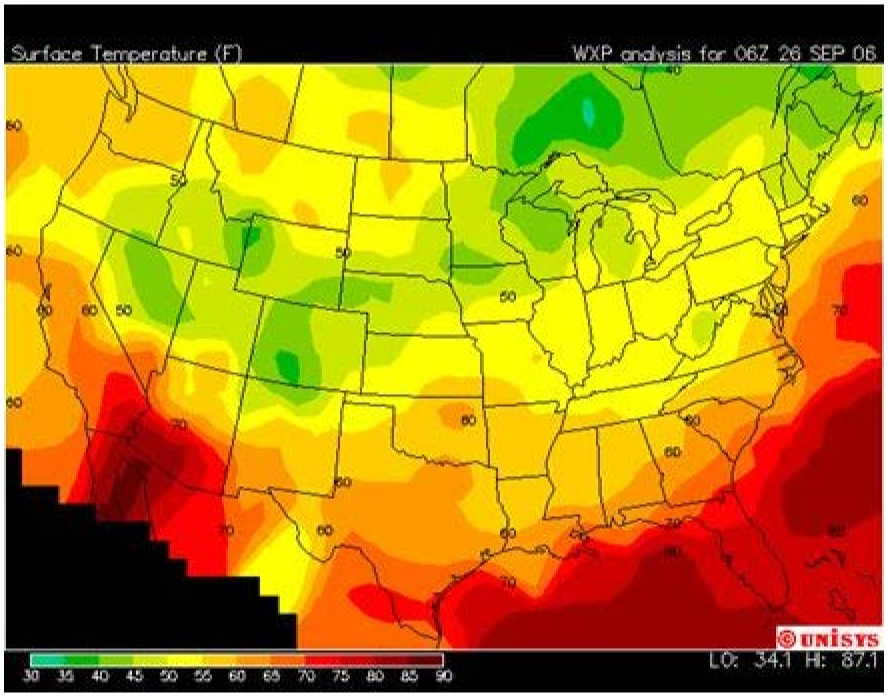
$$\mathbf{v} = U\left[1 + \frac{R^2}{r^2}(\sin^2\theta - \cos^2\theta)\right]\mathbf{i} - U\frac{R^2}{r^2}\cos\theta\sin\theta\,\mathbf{j}$$



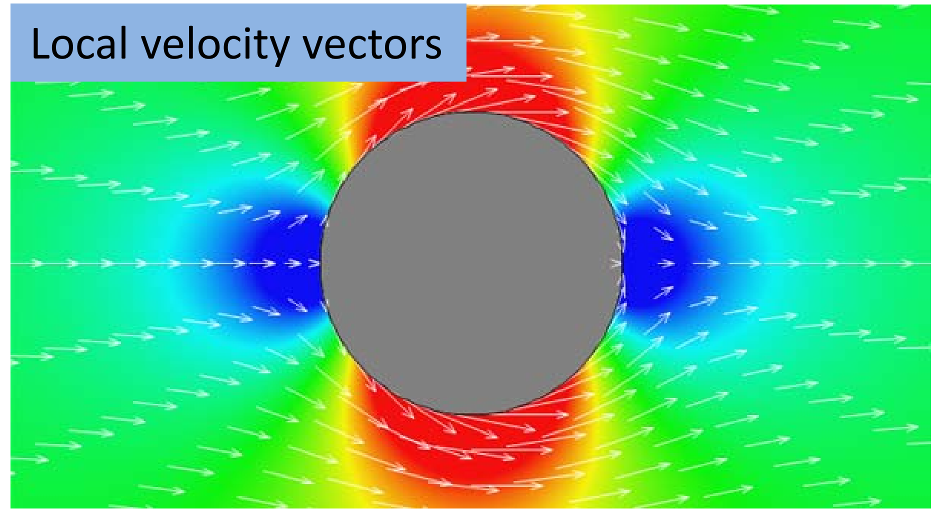
White curves: Contour lines of the velocity potentials $\varphi = const$
Black curves: Field lines of the velocity fields.
In fluid mechanics, field lines of the velocity field are called the **streamlines**.

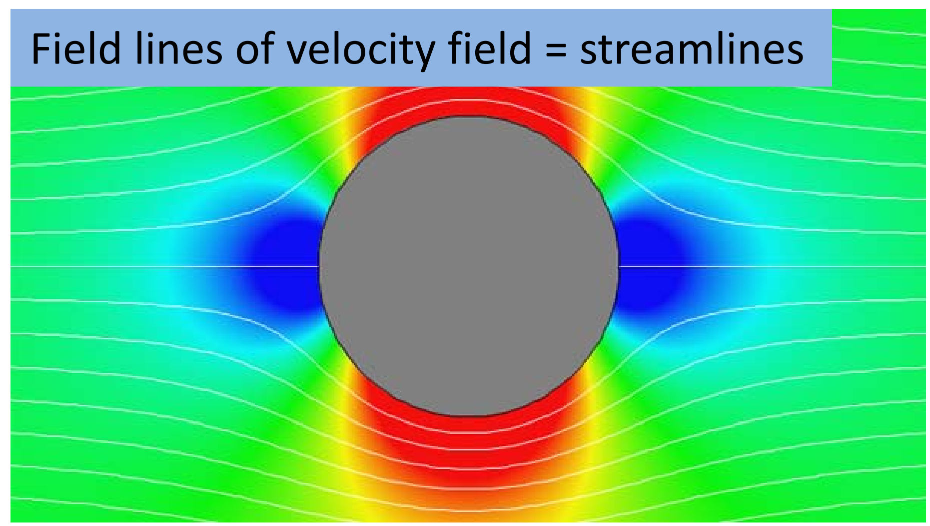
6.4. Scalar and vector fields and their derivatives

Example 2: Scalar temperature field



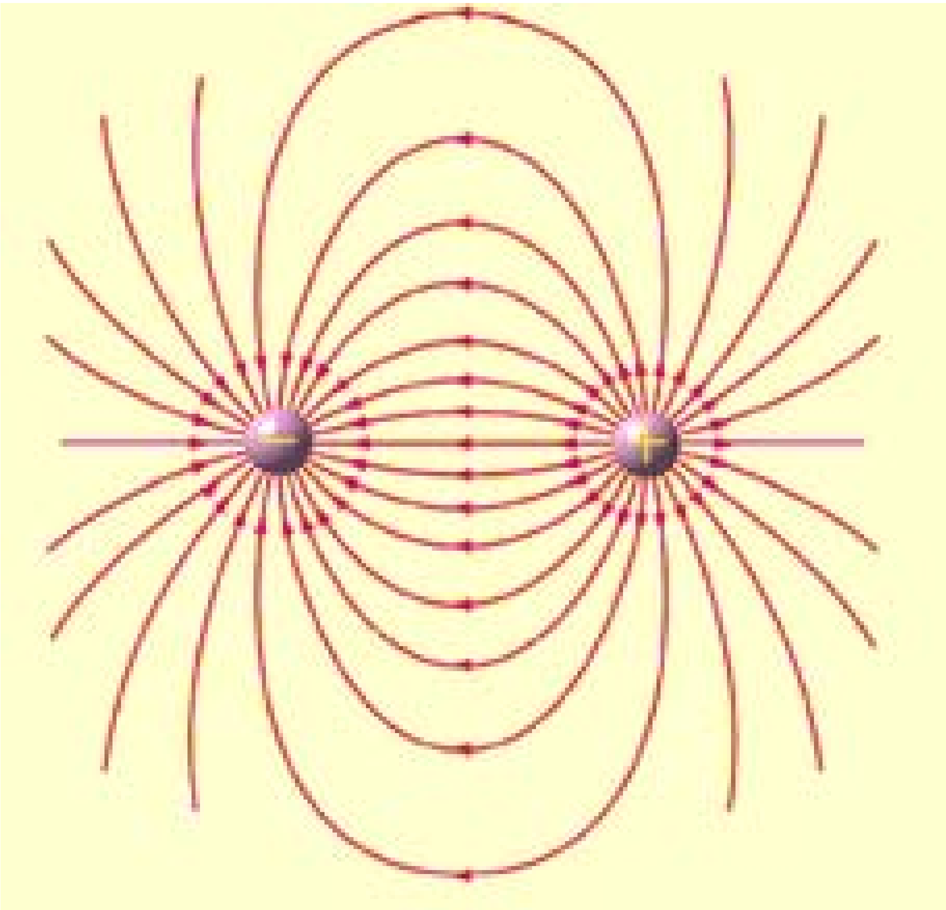
Example 3: Vector velocity field in the flow over a circular cylinder



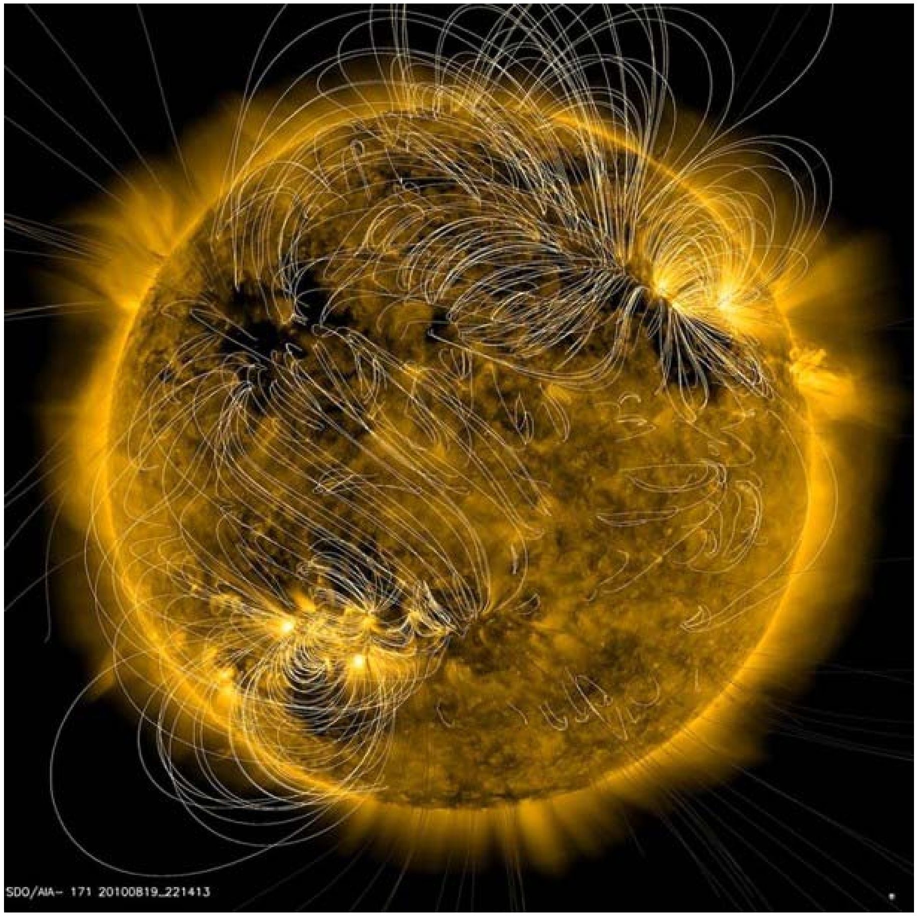


6.4. Scalar and vector fields and their derivatives

Example 4: Electric field lines of a dipole



Example 5: Magnetic field lines over the Sun



6.4. Scalar and vector fields and their derivatives

We can define **derivatives of scalar and vector fields**, e.g., with respect to x

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x} \quad (6.4.3)$$

$$\frac{\partial \mathbf{F}}{\partial x} = \lim_{\Delta x \to 0} \frac{\mathbf{F}(x + \Delta x, y, z) - \mathbf{F}(x, y, z)}{\Delta x} =$$

$$\lim_{\Delta x \to 0} \frac{F_x(x + \Delta x, y, z)\mathbf{i} + F_y(x + \Delta x, y, z)\mathbf{j} + F_z(x + \Delta x, y, z)\mathbf{k} - [F_x(x, y, z)\mathbf{i} + F_y(x, y, z)\mathbf{j} + F_z(x, y, z)\mathbf{k}]}{\Delta x} =$$

$$\frac{\partial F_x}{\partial x}\mathbf{i} + \frac{\partial F_y}{\partial x}\mathbf{j} + \frac{\partial F_z}{\partial x}\mathbf{k} \quad (6.4.4)$$

The basic property of any such derivative is the linearity (a and b are constants):

$$\frac{\partial}{\partial x}(a\mathbf{F} + b\mathbf{G}) = a\frac{\partial \mathbf{F}}{\partial x} + b\frac{\partial \mathbf{G}}{\partial x}$$

Many rules valid for ordinary derivatives also hold for derivatives of the dot and cross products, in particular

$$\frac{\partial}{\partial x}(\mathbf{F} \cdot \mathbf{G}) = \frac{\partial \mathbf{F}}{\partial x} \cdot \mathbf{G} + \mathbf{F} \cdot \frac{\partial \mathbf{G}}{\partial x}$$

$$\frac{\partial}{\partial x}(\mathbf{F} \times \mathbf{G}) = \frac{\partial \mathbf{F}}{\partial x} \times \mathbf{G} + \mathbf{F} \times \frac{\partial \mathbf{G}}{\partial x}$$

These properties remain valid because $\mathbf{F} \cdot \mathbf{G}$ $\mathbf{F} \times \mathbf{G}$ are represented in the form of sums, where every term is a product of two functions, e.g. $\mathbf{F} \cdot \mathbf{G} = F_x G_x + F_y G_y + F_z G_z$.

6.5. Curves

Let's consider a 3D space with Cartesian coordinates $Oxyz$ and a vector $\mathbf{r}$ in this space, which every coordinate is a function of some **parameter** t:

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \qquad a \le t \le b \tag{6.5.1}$$

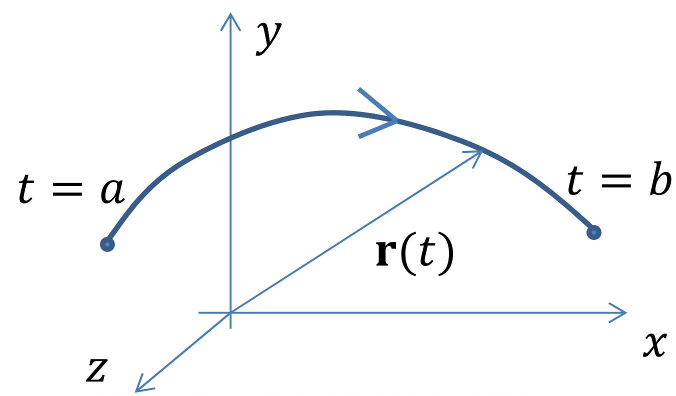
$\mathbf{r}(t)$ is called the **vector function** of parameter t. Functions $x(t)$, $y(t)$, and $z(t)$ are called the **coordinate functions** of the vector function $\mathbf{r}(t)$.

If $\mathbf{r}(t)$ is the position vector, then Eq. (6.5.1) determines a set of point C in 3D space that form a **curve**. In this case Eq. (6.5.1) is called the **parametric representation of a curve**.

Note: In 2D space, e.g., on the plane Oxy, we can use the parametric representation of a **plane (2D) curve** in the form

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \qquad a \le t \le b$$

Plane curve is a particular case of **twisted** (**3D**) curves. Curve C given by Eq. (6.5.1) is called **oriented** since an increase of the parameter t defines the direction of motion along C, $\mathbf{r}(a)$ is the **initial point**, $\mathbf{r}(b)$ is **the terminal point**.



Example 1: Parametric representation of a straight line in 3D

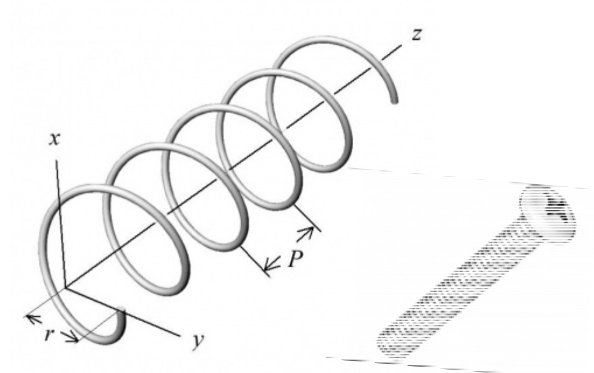
$$\mathbf{r}(t) = \mathbf{r}_0 + (a\mathbf{i} + b\mathbf{j} + c\mathbf{k})t = \mathbf{r}_0 + \mathbf{d}t$$

Example 2: Parametric representation of a circle in 2D

$$\mathbf{r}(t) = \mathbf{r}_0 + R(\cos t\,\mathbf{i} + \sin t\,\mathbf{j})$$

Example 3: Parametric representation of a helix curve

$$\mathbf{r}(t) = r(\cos t\,\mathbf{i} + \sin t\,\mathbf{j}) + (P/2\pi)t\mathbf{k}$$



Simple curve is a curve without multiple points, that is, without points at which the curve intersects or touch itself.

A curve is called **smooth in point** P (or at value t of its parameter), if its coordinate functions have derivatives of any order in P. A curve is called **smooth** if it is smooth in every point of (a, b).

A curve is called **piecewise smooth** if it consists of finitely many smooth curves, i.e. can be represented as a set of a finite number of smooth curves and $a \le t \le b$ can be divided into a set of disjoint subintervals and for every subinterval C_i is smooth.

If curve C is smooth in point t, we can define the vector function

$$\mathbf{r}'_t(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$

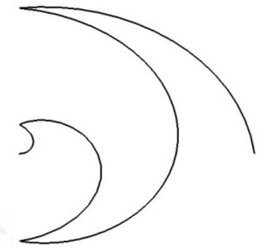
(6.5.2)

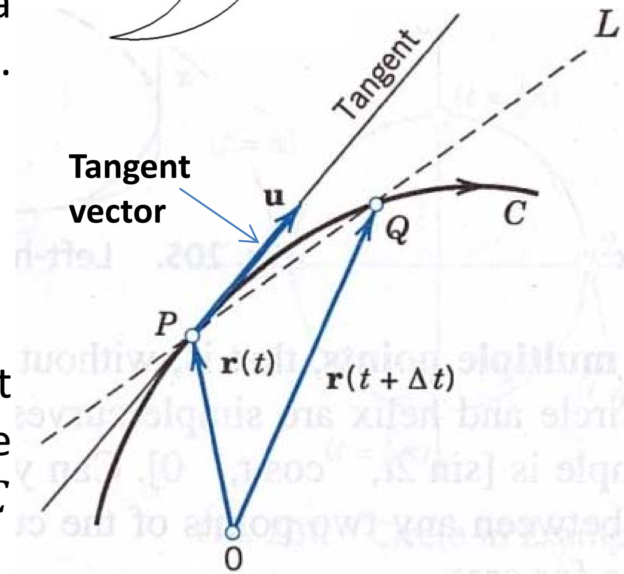
If $|\mathbf{r}'| \neq 0$, then vector $\mathbf{r}'$ takes the limiting position with respect to the chord vector $\mathbf{r}(t + \Delta t) - \mathbf{r}(t)$ along the **tangent** to the curve in point P and is called the **tangent vector** to the curve C in the point P.

Example: If t is time, $\mathbf{r}(t)$ is the trajectory of a point mass, then $\mathbf{r}'_t(t) = \mathbf{v}(t)$ is the velocity of the point mass.

Curves with multiple points

Piecewise smooth curve





The parametric representation of the tangent to the curve is $\mathbf{r}(w) = \mathbf{r}(t) + \mathbf{r}'_t(t)w$

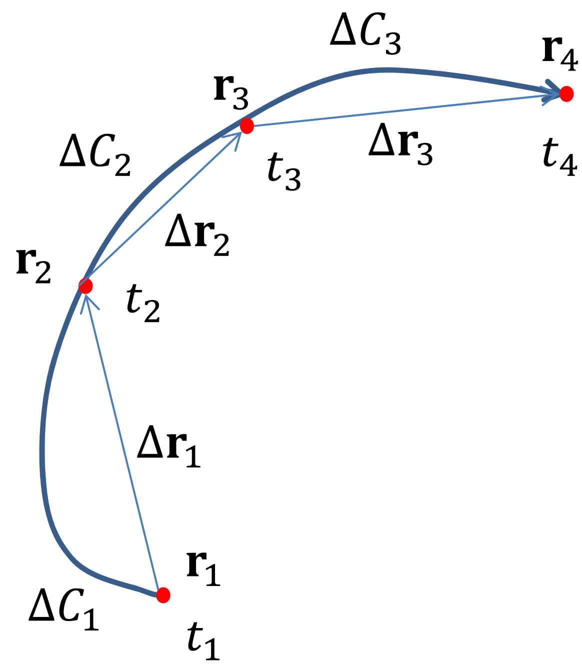
6.5. Curves

Let's consider the **arc** (part of the curve) that corresponds to the interval of the parameter (a, t), $t < b$ and divide the arc by points

$$\mathbf{r}_i = \mathbf{r}(t_i), \qquad t_1 = \text{a} < t_1 < t_1 < \cdots < t_N = t$$

into a large set of curves ΔC_i. For every pair of neighbor points, we can introduce the line segment with the chord vector $\Delta\mathbf{r}_i = \mathbf{r}_{i+1} - \mathbf{r}_i$, which approximates curve ΔC_i.

Let $\Delta r_{max} = \max(|\Delta\mathbf{r}_i|\ )$. The smaller Δr_{max}, the better approximation of the arc by the broken line of chords with endpoints $\mathbf{r}_i$. Then it is reasonable to call the following limit

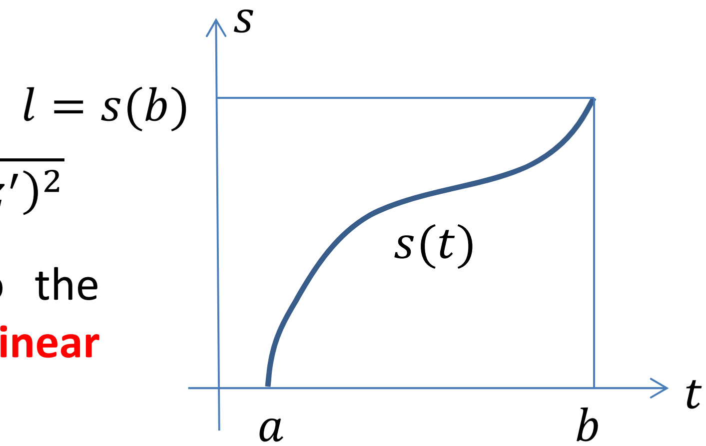


$$s(t) = \lim_{\Delta r_{max}\to 0} \sum_{i=1}^{N-1} |\Delta\mathbf{r}_i| = \lim_{\Delta t_{max}\to 0} \sum_{i=1}^{N-1} \left|\frac{\Delta\mathbf{r}_i}{\Delta t_i}\right| \Delta t_i = \int_a^t |\mathbf{r}'_t| dt \qquad (6.5.3)$$

the **arc length**. $l = s(b)$ is the **curve length**.

Definition (6.5.3) shows us that $ds/dt = |\mathbf{r}'_t|$ and

$$ds = \frac{ds}{dt}dt = |\mathbf{r}'_t|dt, \qquad \frac{ds}{dt} = |\mathbf{r}'_t| = \sqrt{(x')^2 + (y')^2 + (z')^2}$$



i.e. differential dt of the parameter t corresponds to the increment of the arc length $ds = |\mathbf{r}'|dt$ called the **linear element** of C.

6.5. Curves

Since $s(t)$ is an increasing function, then s can be used as another parameter. For this purpose, we can resolve equation $s = s(t)$ with respect to $t = t(s)$. Then :

$$\mathbf{r}(s) = x(t(s))\mathbf{i} + y(t(s))\mathbf{j} + z(t(s))\mathbf{k} = \tilde{x}(s)\mathbf{i} + \tilde{y}(s)\mathbf{j} + \tilde{z}(s)\mathbf{k}, \qquad 0 \le s \le l$$

where $\tilde{x}(s) = x\big(t(s)\big)$, etc. For every curve there is a specific parameterization where parameter s is the arc length. For this parameterization

$$\mathbf{r}_s'(s) = \tilde{x}'\mathbf{i} + \tilde{y}'\mathbf{j} + \tilde{z}'\mathbf{k} = x't'\mathbf{i} + y't'\mathbf{j} + z't'\mathbf{k}$$

where

$$\tilde{x}' = \frac{d\tilde{x}}{ds} = \frac{dx}{dt}\frac{dt}{ds} = x't'$$

$$t' = \frac{dt}{ds} = \left(\frac{ds}{dt}\right)^{-1} = \frac{1}{|\mathbf{r}_t'|}$$

Thus,

$$|\mathbf{r}_s'| = \frac{\sqrt{(x')^2 + (y')^2 + (z')^2}}{|\mathbf{r}_t'|} = 1$$

i.e. *if the parameter is the arc length, the tangent vector is the unit tangent vector* $\mathbf{u} = \mathbf{r}_s'$.

Theorem: If a vector function $\mathbf{r}(t)$ has unit length, $|\mathbf{r}(t)| = 1$, then the tangent vector $\mathbf{r}_t'$ is orthogonal to $\mathbf{r}$, i.e. $\mathbf{r} \cdot \mathbf{r}_t' = 0$.

Proof: Assume $\sqrt{x^2 + y^2 + z^2} = 1$. Then differentiation of the this equation results in:

$$\frac{2xx' + 2yy' + 2zz'}{2\sqrt{x^2 + y^2 + z^2}} = \mathbf{r} \cdot \mathbf{r}_t' = 0$$

6.5. Curves

Here all derivatives are calculated for $t = s$, i.e. $\mathbf{u}' = \mathbf{u}'_s$, $\mathbf{b}' = \mathbf{b}'_s$

Consequence:

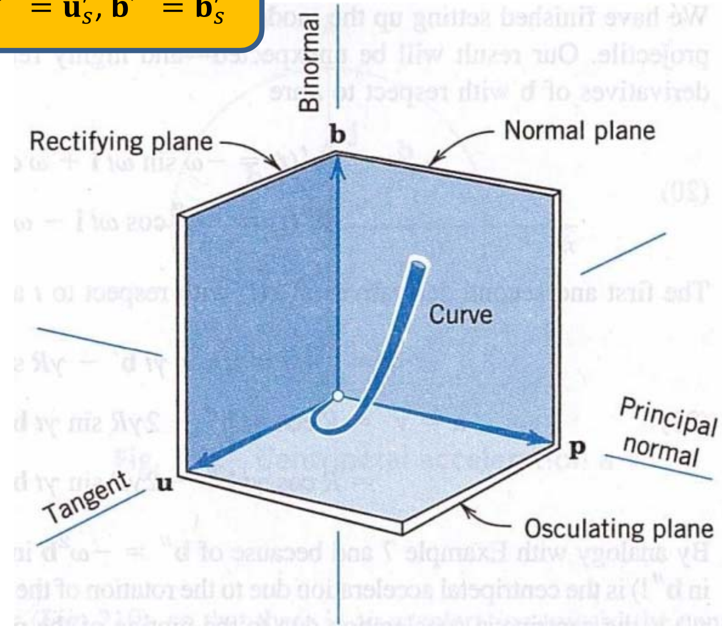
If the parameter is the arc length then

$$\mathbf{r}'_s \cdot \mathbf{r}''_s = \mathbf{u} \cdot \mathbf{u}' = 0$$

If $|\mathbf{u}'| \neq 0$, then $\mathbf{u}'$ is called the **principal normal vector**, $\mathbf{p} = \mathbf{u}'/|\mathbf{u}'|$ is called the **unit principal normal vector**, and $\mathbf{b} = \mathbf{u} \times \mathbf{p}$ is called the **unit binormal vector**.

Vectors $\mathbf{u}$, $\mathbf{p}$, and $\mathbf{b}$ form the **local trihedron** of mutually orthogonal vectors in a point of a curve.

Planes $(\mathbf{u}, \mathbf{p})$, $(\mathbf{p}, \mathbf{b})$, and $(\mathbf{u}, \mathbf{b})$ are called **osculating, normal, and rectifying planes.**



Two major numerical properties of a twisted curve are the **curvature** $\kappa(s)$ and **torsion** $\tau(s)$:

$$\kappa(s) = |\mathbf{u}'| \qquad \tau(s) = -\mathbf{p} \cdot \mathbf{b}' = \pm|\mathbf{b}'| \qquad (6.5.4)$$

Note: $\mathbf{b}'$ is parallel to $\mathbf{p}$, since $\mathbf{b}' = (\mathbf{u} \times \mathbf{p})' = \mathbf{u}' \times \mathbf{p} + \mathbf{u} \times \mathbf{p}' = \mathbf{u} \times \mathbf{p}'$, i.e. $\mathbf{b}'$ is orthogonal to both $\mathbf{u}$ and $\mathbf{b}$.

Meaning of curvature: It shows how fast the curve deviates from its tangent direction in the point P (for a straight line $\mathbf{u} = const$ and $\kappa(s) = 0$; $1/\kappa(s)$ the radius of a circle).

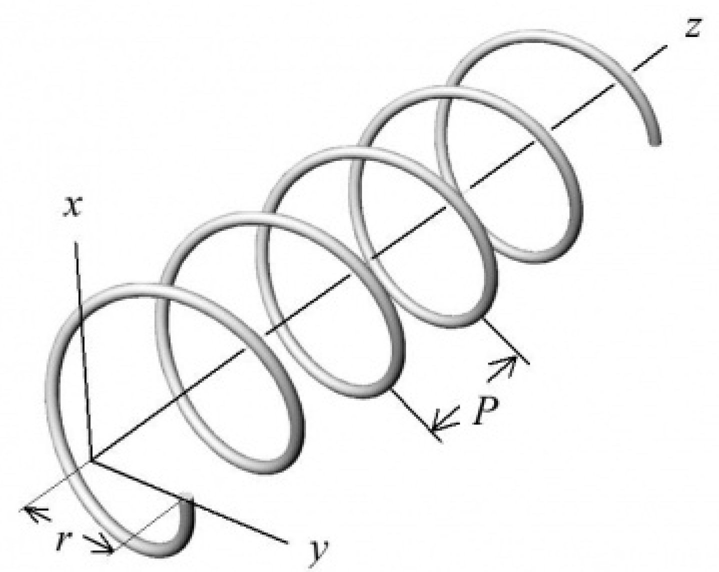
Meaning of torsion: It shows how fast the curve deviates from a plane curve in the osculating plane (for a plane curve $\mathbf{b} = const$ and $\tau(s) = 0$).

Note: With exception of the following example, we will not use the curvature and torsion.

6.5. Curves

Example: Properties of a **helix curve.**

Parametric representation of the helix curve around axis z of radius R and pitch P:



$$\mathbf{r}(t) = R\cos t\,\mathbf{i} + R\sin t\,\mathbf{j} + \frac{Pt}{2\pi}\mathbf{k}$$

$$\mathbf{r}_t'(t) = -R\sin t\,\mathbf{i} + R\cos t\,\mathbf{j} + \frac{P}{2\pi}\mathbf{k}$$

$$\frac{ds}{dt} = |\mathbf{r}_t'| = \sqrt{R^2 + \left(\frac{P}{2\pi}\right)^2} = \alpha = const \qquad \Longrightarrow s = \alpha t$$

$$\mathbf{r}(s) = R\cos\frac{s}{\alpha}\mathbf{i} + R\sin\frac{s}{\alpha}\mathbf{j} + \frac{Ps}{2\pi\alpha}\mathbf{k}$$

$$\mathbf{u} = \mathbf{r}_s'(s) = -\frac{R}{\alpha}\sin\frac{s}{\alpha}\mathbf{i} + \frac{R}{\alpha}\cos\frac{s}{\alpha}\mathbf{j} + \frac{P}{2\pi\alpha}\mathbf{k}$$

$$\mathbf{u}' = -\frac{R}{\alpha^2}\cos\frac{s}{\alpha}\mathbf{i} - \frac{R}{\alpha^2}\sin\frac{s}{\alpha}\mathbf{j}$$

$$\mathbf{p} = \frac{\mathbf{u}'}{|\mathbf{u}'|} = -\cos\frac{s}{\alpha}\mathbf{i} - \sin\frac{s}{\alpha}\mathbf{j}$$

$$\mathbf{b} = \mathbf{u}\times\mathbf{p} = \frac{P}{2\pi\alpha}\sin\frac{s}{\alpha}\mathbf{i} - \frac{P}{2\pi\alpha}\cos\frac{s}{\alpha}\mathbf{j} + \frac{R}{\alpha}\mathbf{k}$$

$$\mathbf{b}' = \frac{P}{2\pi\alpha^2}\cos\frac{s}{\alpha}\mathbf{i} + \frac{P}{2\pi\alpha^2}\sin\frac{s}{\alpha}\mathbf{j}$$

Curvature and torsion of the helix curve:

$$\kappa(s) = |\mathbf{u}'| = \frac{R}{\alpha^2} = \frac{R}{R^2 + \left(\frac{P}{2\pi}\right)^2} = \frac{1}{R}\frac{1}{1 + \left(\frac{P}{2\pi R}\right)^2}, \qquad \tau(s) = \frac{P/2\pi}{\alpha^2}$$

6.6. Surfaces

The **surface** in 3D space is a set of points that can be given by an equation

$$G(x, y, z) = 0 \quad (6.6.1)$$

Example: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$ is the sphere of radius R with the center in the point (x_0, y_0, z_0).

Locally, we can resolve Eq. (6.6.1) with respect to one of coordinates x, y, z and obtain, e.g.

$$x = g(y, z) \quad (6.6.2)$$

For instance, at any point of the sphere, where $x > x_0$, we can write

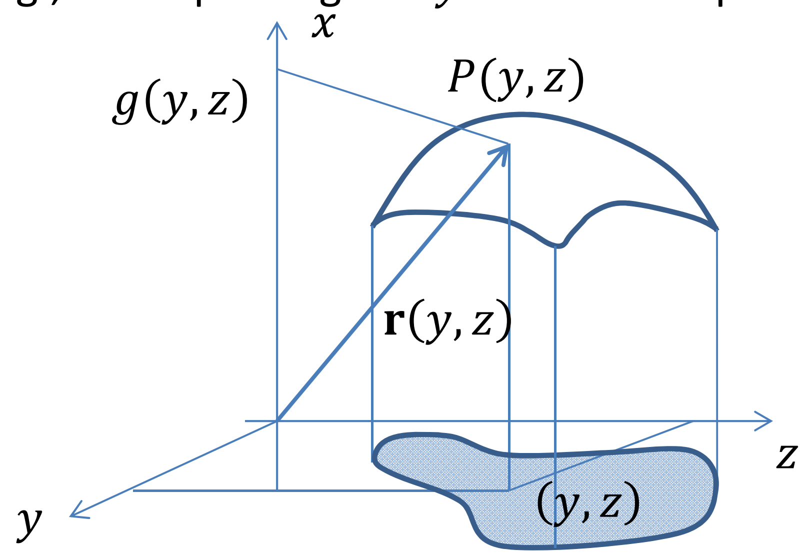
$$x = x_0 + \sqrt{R^2 - (y - y_0)^2 + (z - z_0)^2}$$

Equation in the form (6.6.2) may be not valid, however, for the whole surface, since given y and z may correspond to a few different x at the surface (e.g., for a sphere given y and z correspond to two points on the sphere surface).

On the other hand, Eq. (6.6.2) shows that locally the surface can be represented parametrically as

$$\mathbf{r}(y, z) = g(y, z)\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad (6.6.3)$$

where $\mathbf{r}(y, z)$ denotes a radius vector of a point at the surface. Eq. (6.6.3) shows that a surface is a two-dimensional set of points, since we need two parameters (y and z) to describe it parametrically.



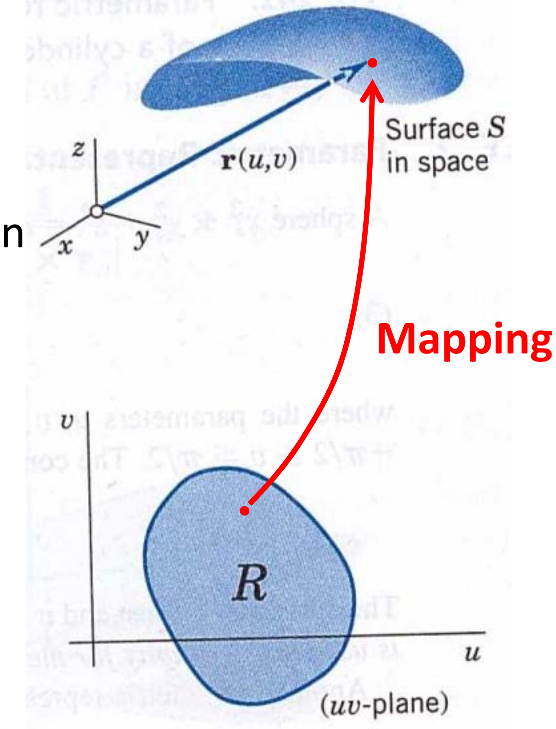
6.6. Surfaces

In general, parameters do not need to coincide with Cartesian coordinates, so one can introduce the general **parametric representation** of a surface in the form

(6.6.4) $$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

where u and v vary in some region R on uv-plane.

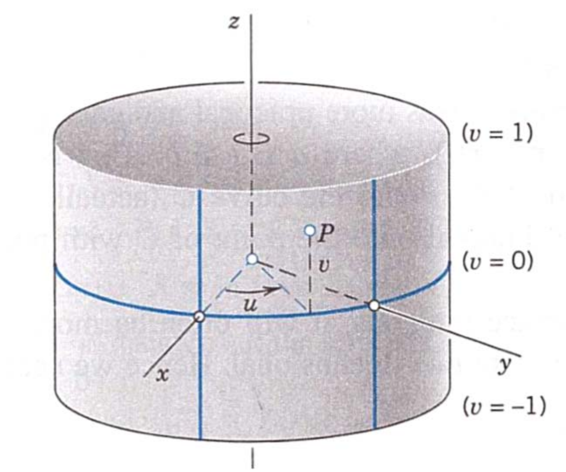
Eq. (6.6.4) describes the parametric **mapping** of the plane region R in 3D surface.



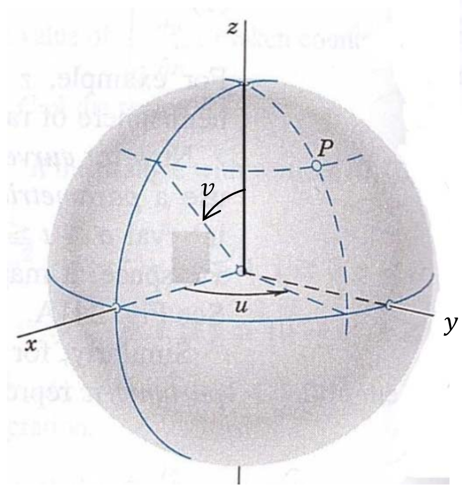
In order to define any surface parametrically, we need to

1. Introduce three functions: $x(u,v)$, $y(u,v)$, and $z(u,v)$.
2. Define region R on uv-plane where u and v varies.

Example: Parametric representation of a cylinder and a sphere



$$0 \le u \le 2\pi$$
$$a \le v \le b$$



$$0 \le u \le 2\pi$$
$$0 \le v \le \pi$$

$$\mathbf{r} = R\,(\cos u\,\mathbf{i} + \sin u\;\mathbf{j}) + v\mathbf{k}$$

$$\mathbf{r} = R\,(\sin v \cos u\,\mathbf{i} + \sin v \sin u\;\mathbf{j} + \cos v\,\mathbf{k})$$

6.6. Surfaces

If we fix either u or v, Eq. (6.6.4) gives the parametric representation of a curve. Thus, the parametric representation (6.6.4) defines two families of curves on the surface:

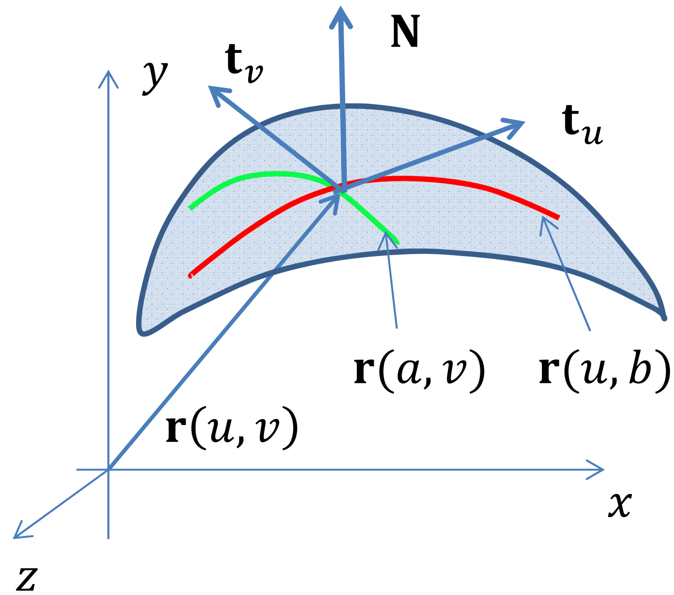
***u*-curves** $(v = b = const)$:

$$\mathbf{r}_u(u) = x(u,b)\mathbf{i} + y(u,b)\mathbf{j} + z(u,b)\mathbf{k}$$

***v*-curves** $(u = a = const)$:

$$\mathbf{r}_v(v) = x(a,v)\mathbf{i} + y(a,v)\mathbf{j} + z(a,v)\mathbf{k}$$

u- and v-curves are called the **grid lines**.

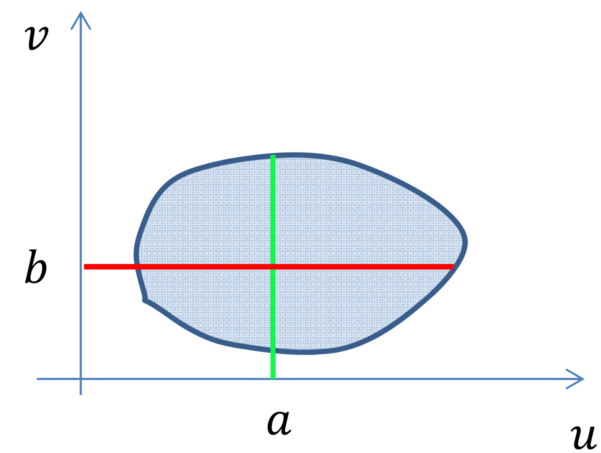


If both $\mathbf{r}(u)$ and $\mathbf{r}(v)$ have derivatives over u and v in point P, then we can introduce in this point two vectors

(6.6.5) $$\mathbf{t}_u = \frac{d\mathbf{r}_u}{du} = \frac{\partial \mathbf{r}(u,v)}{\partial u}, \qquad \mathbf{t}_v = \frac{d\mathbf{r}_v}{dv} = \frac{\partial \mathbf{r}(u,v)}{\partial v}$$

These vectors are tangents in point P to the corresponding grid lines.

If $\mathbf{t}_u \times \mathbf{t}_v \neq 0$, then we can plot a unique plane, containing the point P and vectors $\mathbf{t}_u$ and $\mathbf{t}_u$. This plane is called the **tangent plane** to the surface in point P.



If tangent plane exists in point P, then vector $\mathbf{N} = \mathbf{t}_u \times \mathbf{t}_v$ is called the **normal** to the surface at point P ($\mathbf{N}$ is normal to both $\mathbf{t}_u$ and $\mathbf{t}_v$). Vector $\mathbf{n} = \mathbf{N}/|\mathbf{N}| = \mathbf{t}_u \times \mathbf{t}_v / |\mathbf{t}_u \times \mathbf{t}_v|$ is called the **unit normal**.

6.6. Surfaces

Surface is called **smooth in a point** P if in this point the surface has a unique tangent plane (i.e. $\mathbf{t}_u$ and $\mathbf{t}_v$ exist and $|\mathbf{t}_u \times \mathbf{t}_v| \neq 0$).
Surface is called **smooth** if it is smooth in every its point.
Surface is called **piecewise smooth** if it can be represented as a set of adjoin smooth surfaces.

Physical meaning of the tangent plane: If we perform calculation of surface properties in an infinitesimal vicinity of point P, we can perform all calculation on the tangent plane. Let's use this rule in order to calculate the area of the surface that corresponds to increments du and dv.

Let's assume that we consider point P at the surface when parameters are equal to u and v and we give them infinitesimal increments du and dv. These increments correspond to the surface element of infinitesimal area dA. **Question**: How is dA related to du and dv?
On the tangent plane, du and dv correspond to the parallelogram of area

(6.6.6) $$dA = |(\mathbf{t}_u du) \times (\mathbf{t}_v dv)| = |\mathbf{t}_u \times \mathbf{t}_v| du dv = |\mathbf{N}| du dv$$

For the future consideration, we will also need areas of projections of this parallelogram onto coordinate planes (x, y), (x, z), and (y, z). Let's introduce the **direction cosines** for the unit normal $\mathbf{n} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$**,** where α is an angle between normal and x-axis ($\mathbf{n} \cdot \mathbf{i} = \cos\alpha$), etc. Then the are of projection of dA onto (y, z), $dydz$, is equal to

$$dydz = \cos\alpha\, dA = (\mathbf{n} \cdot \mathbf{i}) dA = \frac{(\mathbf{t}_u \times \mathbf{t}_v) \cdot \mathbf{i}}{|\mathbf{t}_u \times \mathbf{t}_v|} |\mathbf{t}_u \times \mathbf{t}_v| du dv = |(\mathbf{t}_u \times \mathbf{t}_v) \cdot \mathbf{i}| du dv$$

(6.6.7) $$dxdz = \cos\beta\, dA = (\mathbf{n} \cdot \mathbf{j}) dA = |(\mathbf{t}_u \times \mathbf{t}_v) \cdot \mathbf{j}| du dv$$

$$dxdy = \cos\gamma\, dA = (\mathbf{n} \cdot \mathbf{k}) dA = |(\mathbf{t}_u \times \mathbf{t}_v) \cdot \mathbf{k}| du dv$$

The parametric representation

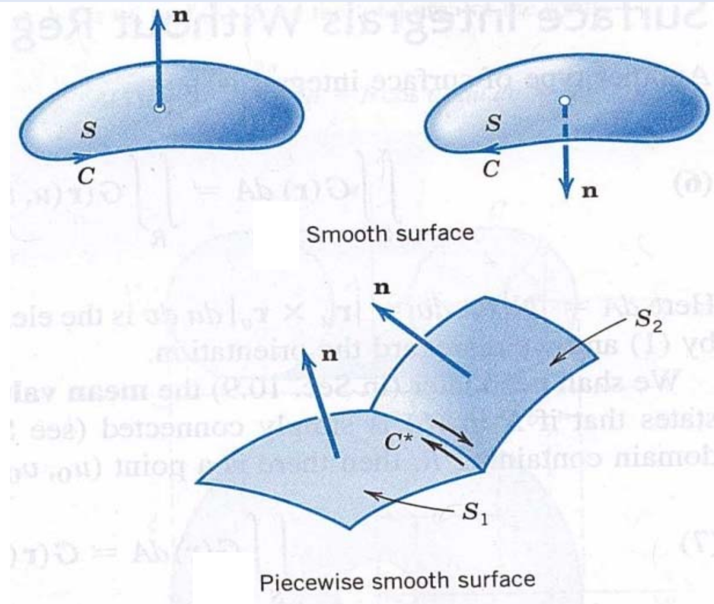
$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

actually define two surfaces that are different in every point by the direction of the normal: one surface has normal $\mathbf{N} = \mathbf{t}_u \times \mathbf{t}_v$, another has normal $-\mathbf{N}$.

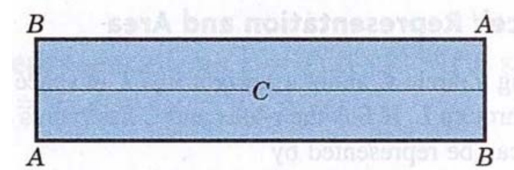
If we choose a unique direction of the normal in every point of the surface, we say we **choose orientation** of the surface.

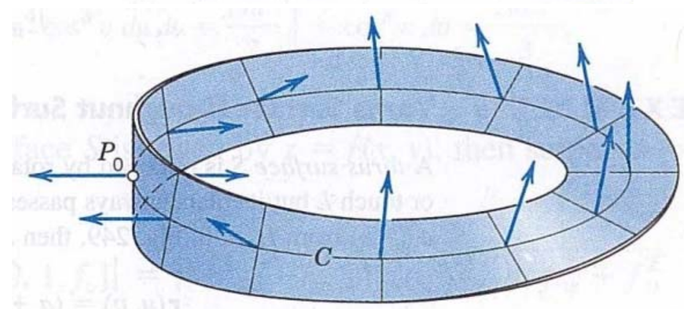
A smooth surface S is called **orientable** if the normal direction given at an arbitrary point of S can be continued in a unique and continuous way to the entire surface.

A piecewise smooth surface S is called **orientable** if we can orient every smooth piece of S so that along each curve C_*, which is a common boundary of two pieces S_1 and S_2, the positive direction of C_* relative to S_1 is opposite to the direction of C_* relative to S_2.



Example: Möbius strip is the nonorientable surface:

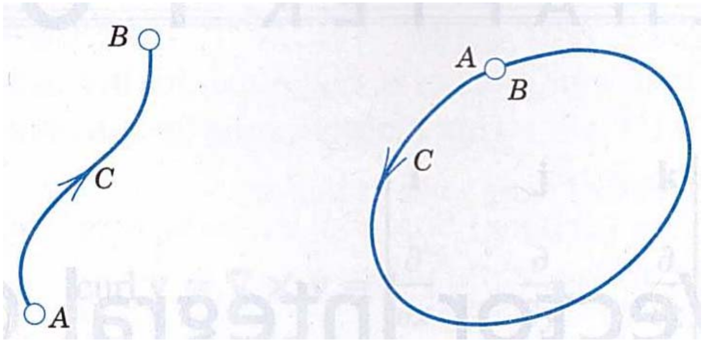




6.7. Line integrals of scalar and vector fields

Let's consider an oriented curve C given by the parametric representation:

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \qquad a \le t \le b$$



We assume that C is a smooth curve, so $\mathbf{r}' = d\mathbf{r}/dt$ exists in a any point of C and the linear element is $ds = \sqrt{x'^2 + y'^2 + z'^2}dt$.

The **line integral of a scalar field** $f(\mathbf{r})$ over the curve C is the integral

$$\int_C f(\mathbf{r})ds \equiv \int_a^b f(x(t), y(t), z(t))\sqrt{x'^2 + y'^2 + z'^2}dt \qquad (6.7.1)$$

The **line integral of a vector field** $\mathbf{F}(\mathbf{r})$ over the curve C is the integral

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b (F_x dx + F_y dy + F_z dz) \equiv \int_a^b \left(F_x x' + F_y y' + F_z z'\right)dt = \int_a^b \mathbf{F} \cdot \mathbf{r}_t' dt \qquad (6.7.2)$$

The curve C is called the **path of integration**. If $\mathbf{r}(a) = \mathbf{r}(b)$, then C is the **closed path** and the line integral of a vector field is called **circulation** of the vector field and denoted as $\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$.

6.7. Line integrals of scalar and vector fields

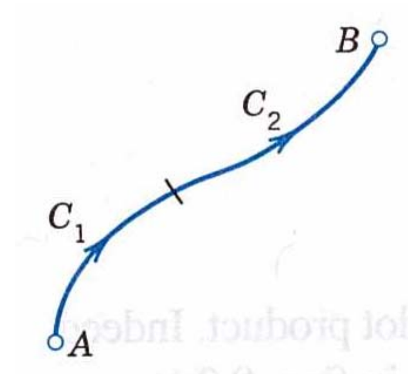
Note: Direction that corresponds to increasing parameter t is called **positive**.

Example: Let's consider motion of a point mass in a force field $\mathbf{F}(\mathbf{r})$ (e.g. gravitational field). Then the line integral over the trajectory $A = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C \mathbf{F}(\mathbf{r}) \cdot \mathbf{v}dt$ is the work done by force $\mathbf{F}$.

Properties of the line integral:

1. Linearity

$$\int_C (a\mathbf{F} + b\mathbf{G}) \cdot d\mathbf{r} = a \int_C \mathbf{F} \cdot d\mathbf{r} + b \int_C \mathbf{G} \cdot d\mathbf{r}$$

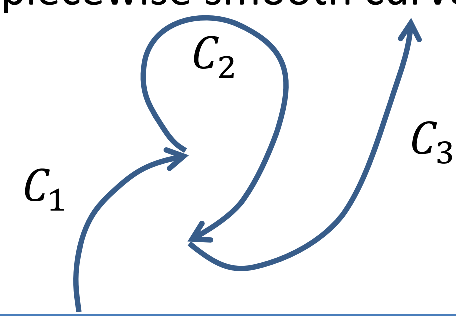


2. Additivity

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

3. Additivity allows one to extent definition of the line integral to any piecewise smooth curve

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \sum_i \int_{C_i} \mathbf{F} \cdot d\mathbf{r}$$



6.7. Line integrals of scalar and vector fields

Theorem:

For a given vector field $\mathbf{F}$ and curve C, all parametric representations of curve C that give the same positive direction on C yield the same value of the line integral.

Proof:

Let's consider another parameterization of curve C with some parameter s: $\mathbf{r}^*(s) = x^*(s)\mathbf{i} + y^*(s)\mathbf{j} + z^*(s)\mathbf{k}$, $a^* \le s \le b^*$. Since it is the representation of the same curve, $s = s(t)$. Then

$$\int_C \mathbf{F}(\mathbf{r}^*) \cdot d\mathbf{r}^* = \int_{a^*}^{b^*} (F_x x^{*\prime} + F_y y^{*\prime} + F_z z^{*\prime}) ds = \int_a^b (F_x x^{*\prime} + F_y y^{*\prime} + F_z z^{*\prime}) \frac{ds}{dt} dt$$

$$= \int_a^b (F_x x' + F_y y' + F_z z')\, dt = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

$$x^{*\prime} \frac{ds}{dt} = x'$$

Example 1: $\mathbf{F} = 0\mathbf{i} + x\mathbf{j} + 0\mathbf{k}$. Let's calculate line integral over a straight path.

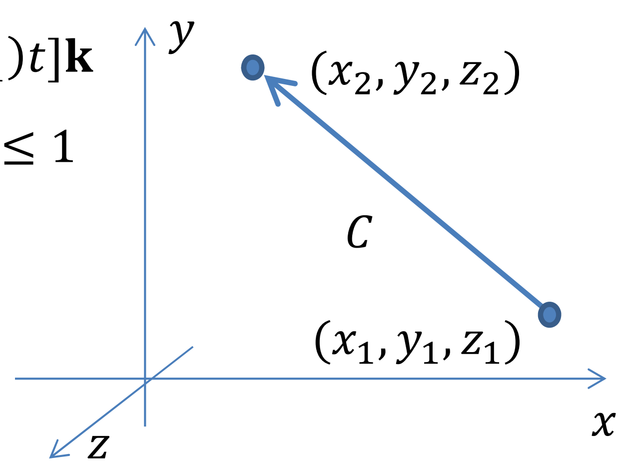
Parametric representation of the path:

$\mathbf{r}(t) = [x_1 + (x_2 - x_1)t]\mathbf{i} + [y_1 + (y_2 - y_1)t]\mathbf{j} + [z_1 + (z_2 - z_1)t]\mathbf{k}$

$\mathbf{r}' = (x_2 - x_2)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$ $\quad 0 \le t \le 1$

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^1 x y' dt = \int_0^1 [x_1 + (x_2 - x_1)t](y_2 - y_1) dt$$

$$= \frac{x_2 + x_1}{2}(y_2 - y_1)$$



6.7. Line integrals of scalar and vector fields

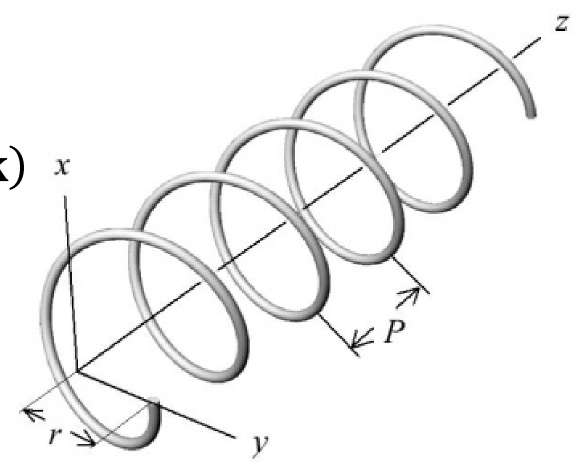
Example 2: $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$, Line integral over the **helix curve**.

Parametric representation of a helix curve around axis z:

Radius $r = 1$

Pitch $P = 6\pi$

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 3t\mathbf{k}, \qquad 0 \le t \le 2\pi$$
$$\mathbf{r}_t{}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 3\mathbf{k}$$
$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}_t{}'(t) = (3t\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}) \cdot (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 3\mathbf{k})$$
$$= -3t\sin t + \cos^2 t + 3\sin t$$



$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^{2\pi} (-3t\sin t + \cos^2 t + 3\sin t)dt = 7\pi$$

Example 3: If $f = 1$, then $l = \int_C f(\mathbf{r})ds$ is the length of curve C.

Example 4: if $f(x, y, z) = \rho r^2 = \rho(x^2 + y^2)$, where ρ is the linear density of the curve (mass per unit length) and $r = \sqrt{x^2 + y^2}$ is the distance from the point to the axis Oz, then

$$I_z = \int_C \rho(x^2 + y^2)ds$$

is the **moment of inertia** of curve C about z-axis.

6.8. Gradient, Divergence, and Curl

Gradient

Let's consider a scalar field $f(x, y, z) = f(\mathbf{r})$. **Gradient** of this field is the vector field grad f:

$$\mathrm{grad}\, f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \qquad (6.8.1)$$

Calculation of grad f can be considered as an application of the **differential operator** ∇ (nabla)

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \qquad (6.8.2)$$

to the scalar field f. Differential operator ∇ can be viewed as a symbolic vector.

Let's consider a 3D scalar field and define some **directional vector** $\mathbf{d} = d_x\mathbf{i} + d_y\mathbf{j} + d_z\mathbf{k}$ of unit length ($|\mathbf{d}| = 1$) and $f(s) = f(\mathbf{r} + s\mathbf{d})$. Then the **directional derivative** of the scalar field $f(\mathbf{r})$ in the direction of vector $\mathbf{d}$ is

$$D_{\mathbf{d}} f = \frac{\partial f}{\partial \mathbf{d}} \equiv \lim_{s\to 0}\frac{f(\mathbf{r} + s\mathbf{d}) - f(\mathbf{r})}{s} = \lim_{s\to 0}\frac{f(s) - f(0)}{s} = \left.\frac{\partial f(s)}{\partial s}\right|_{s=0}$$

The directional derivative indicates the rate of change of the scalar field f in the direction of vector $\mathbf{d}$. Let's calculate $D_{\mathbf{d}} f$ through partial derivatives $\partial f/\partial x$ etc. using the chain rule:

$$D_{\mathbf{d}} f = \frac{\partial f}{\partial x}d_x + \frac{\partial f}{\partial y}d_y + \frac{\partial f}{\partial z}d_z = \nabla f \cdot \mathbf{d} \qquad (6.8.3)$$

6.8. Gradient, Divergence, and Curl

Note: Gradient and directional derivative are easily applied to 2D fields:

$$\text{grad}\, f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}, \qquad D_{\mathbf{d}} f = \frac{\partial f}{\partial x} d_x + \frac{\partial f}{\partial y} d_y = \nabla f \cdot \mathbf{d} \qquad (6.8.4)$$

Properties of gradient and directional derivatives:

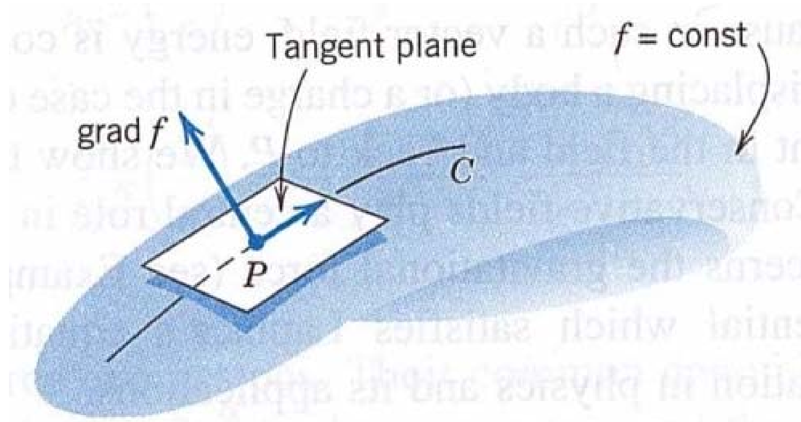
1. *Direction of* ∇f *indicates the direction of the maximum increase of the scalar field* f. The direction of the maximum increase is given by such $\mathbf{d}$ for which $D_{\mathbf{d}} f$ is maximum. From definition of the dot product

$$D_{\mathbf{d}} f = \nabla f \cdot \mathbf{d} = |\nabla f||\mathbf{d}| \cos\gamma = |\nabla f| \cos\gamma$$

where γ is the angle between ∇f and $\mathbf{d}$. $D_{\mathbf{d}} f$ is maximum, when $\gamma = 0$, i.e. $\mathbf{d}$ is directed along ∇f.

2. If we introduce an isosurface in point (x_0, y_0, z_0), which is defined by the equation $f(x, y, z) = f(x_0, y_0, z_0) = c = const$, then ∇f *in this point is directed along the surface normal vector.*

In order to prove it, let's introduce a curve at the isosurface going through point (x_0, y_0, z_0), which is given by the vector function $\mathbf{r}_c(s) = x_c(s)\mathbf{i} + y_c(s)\mathbf{j} + z_c(s)\mathbf{k}$. Since this curve lies on the isosurface, it should satisfy



The **surface normal vector** is perpendicular to the surface tangent plane.

6.8. Gradient, Divergence, and Curl

$$f(x_c(s), y_c(s), z_c(s)) = c$$

Let's differentiate this equation with respect to s in point (x_0, y_0, z_0). Then we will have

$$\frac{\partial f}{\partial x}\frac{dx_c}{ds} + \frac{\partial f}{\partial y}\frac{dy_c}{ds} + \frac{\partial f}{\partial z}\frac{dz_c}{ds} = 0 \quad \Longrightarrow \quad \nabla f \cdot \mathbf{t} = 0$$

where $\mathbf{t} = dx_c/ds\mathbf{i} + dy_c/ds\mathbf{j} + dz_c/ds\mathbf{k}$ is the tangent to the curve which lies in the tangent place to the surface. *Thus, ∇f is orthogonal to any vector lying in the tangent plane, i.e. it is the surface normal vector.*

3. *Gradient is the physical vector, which retains its direction and length after any coordinate transformation*. The gradient is normal to the isosurface and its absolute value is equal to the maximum increase of the scalar field. Since these properties of the scalar field $f(\mathbf{r})$ do not depend on coordinates, the direction and absolute value of ∇f also do not depend on the choice of coordinates.

Example 1: Potential energy and force of the spherically symmetric gravitational field.

The potential energy of a point mass m in the spherically symmetric field of mass M with center in a point (x_0, y_0, z_0) is equal to

$$U(x, y, z) = -GMm/r, \quad r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

Then

$$\nabla U = -\mathbf{F} = \frac{GMm}{r^2}\left(\frac{x - x_0}{r}\mathbf{i} + \frac{y - y_0}{r}\mathbf{j} + \frac{z - z_0}{r}\mathbf{k}\right)$$

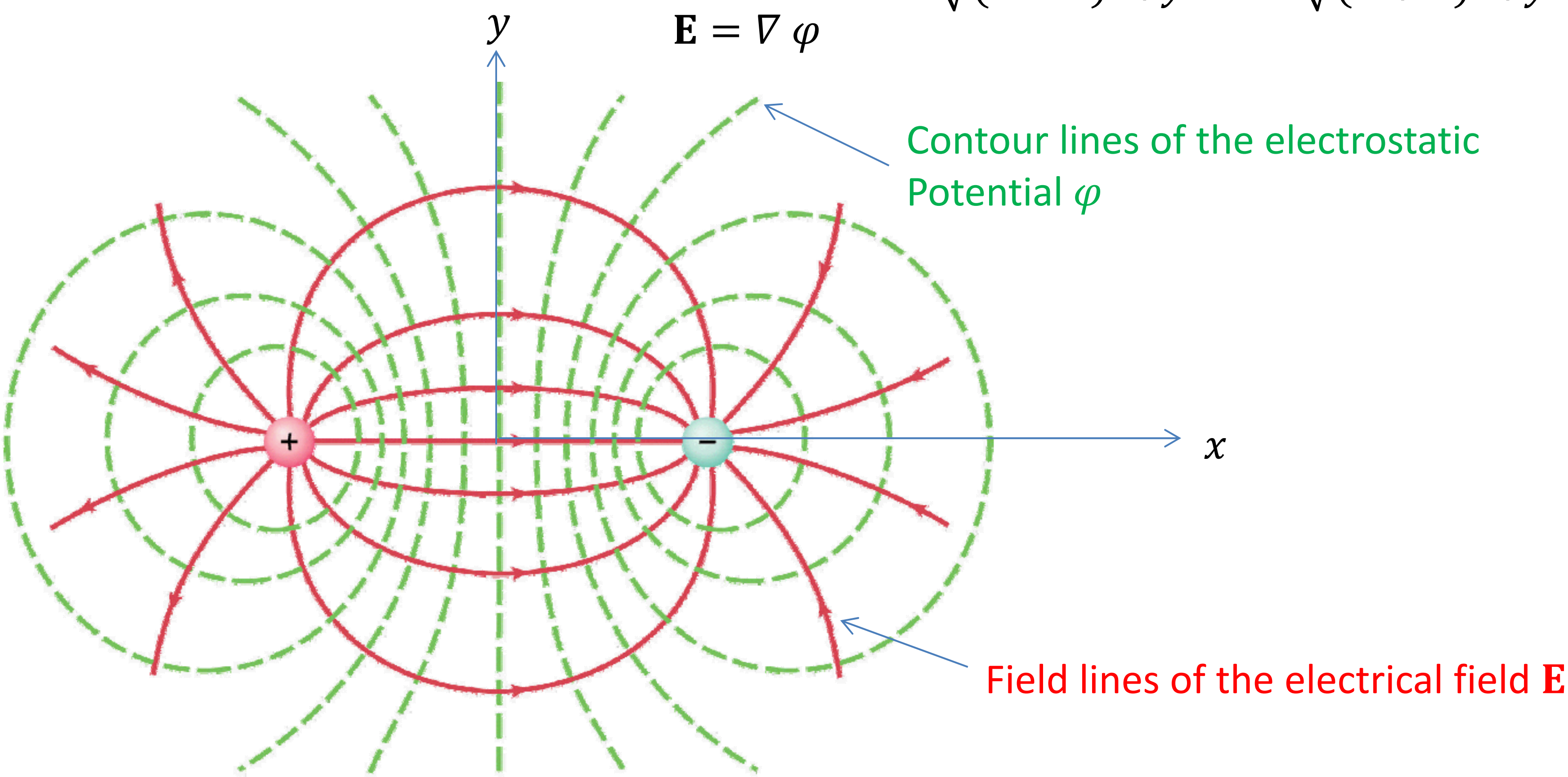
6.8. Gradient, Divergence, and Curl

Note: In 2D, the contour lines of the scalar field $f(x, y) = const$ and field lines of its gradient ∇f constitute two families of curves that are locally normal to each other in every point.

Example 2: **Electric field of a dipole**

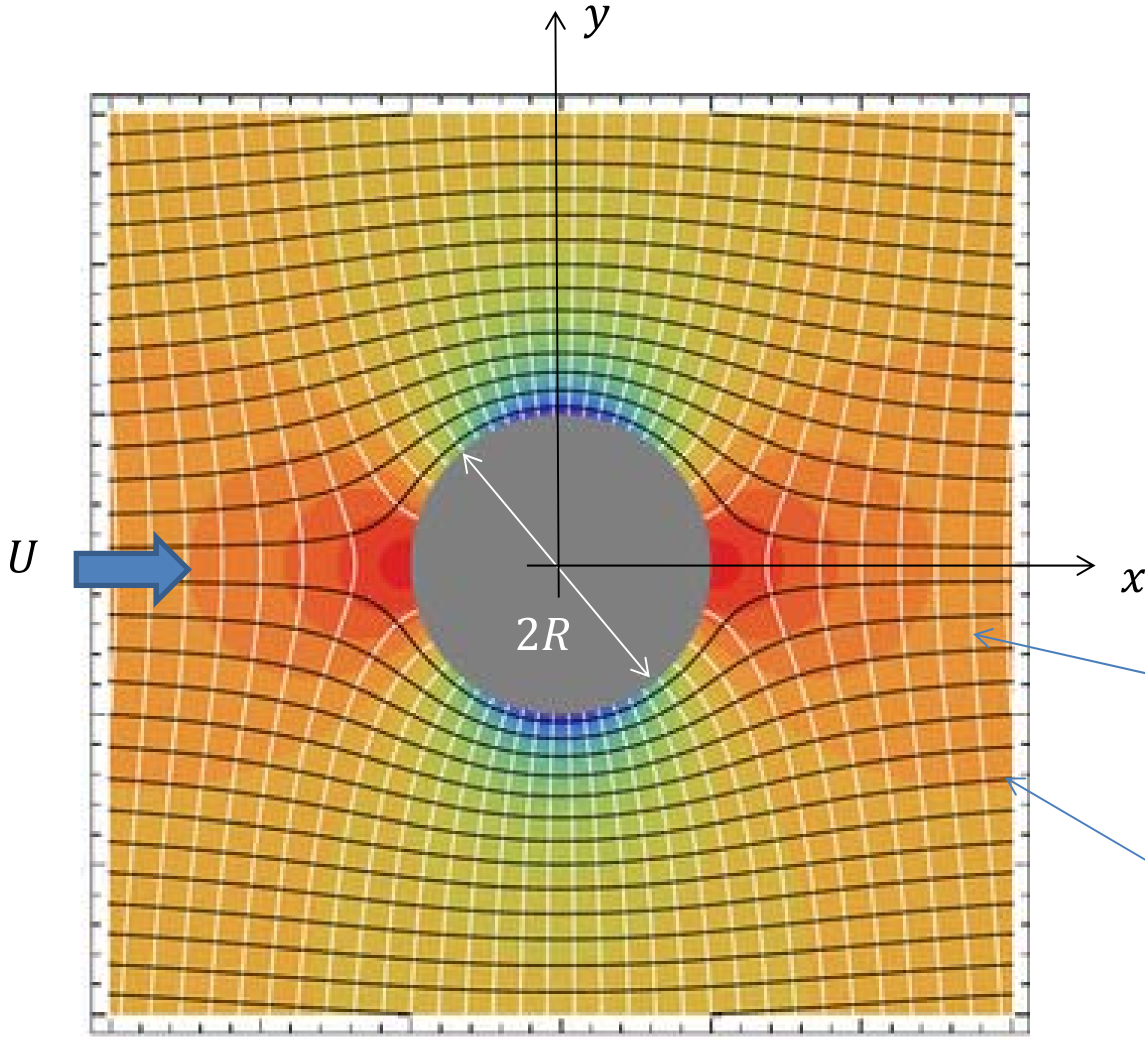
$$\varphi(x, y) = -k_e \frac{q}{\sqrt{(x-a)^2+y^2}} + k_e \frac{q}{\sqrt{(x+a)^2+y^2}}$$

$$\mathbf{E} = \nabla \varphi$$



Note: Two families of locally orthogonal curves can be used in order to introduce non-Cartesian *orthogonal* coordinates in 2D.

Example 3: Potential fluid flow over a cylinder in the cross flow



$$x = r\cos\theta,\ y = r\sin\varphi$$

$$\varphi(r,\theta) = U\left(r + \frac{R^2}{r}\right)\cos\theta$$

$$\mathbf{v} = \nabla\varphi$$

Contour lines of the flow potential φ (white)

Field lines of the velocity field $\mathbf{v}$ (**streamlines**; black)

These curves can be used in order to introduce orthogonal coordinates.

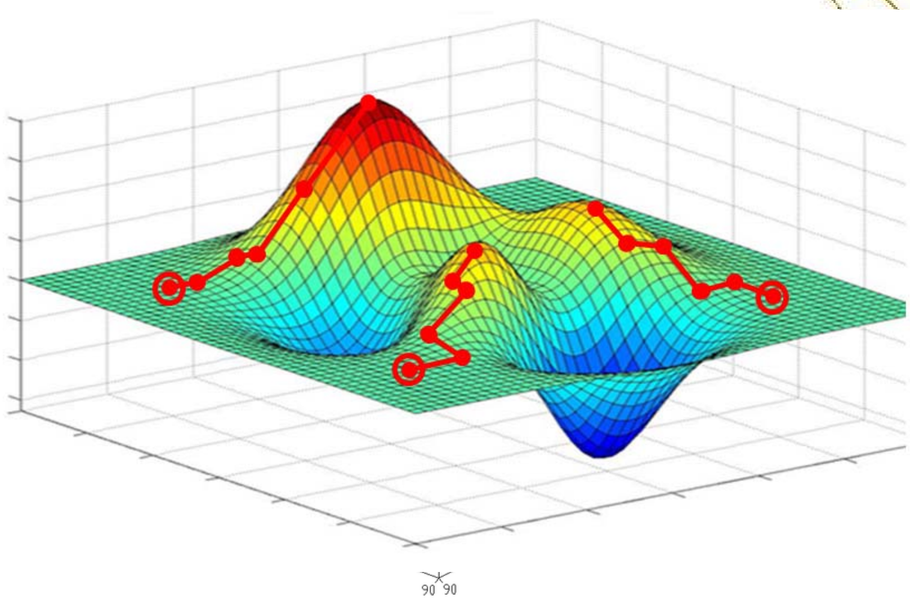
6.8. Gradient, Divergence, and Curl

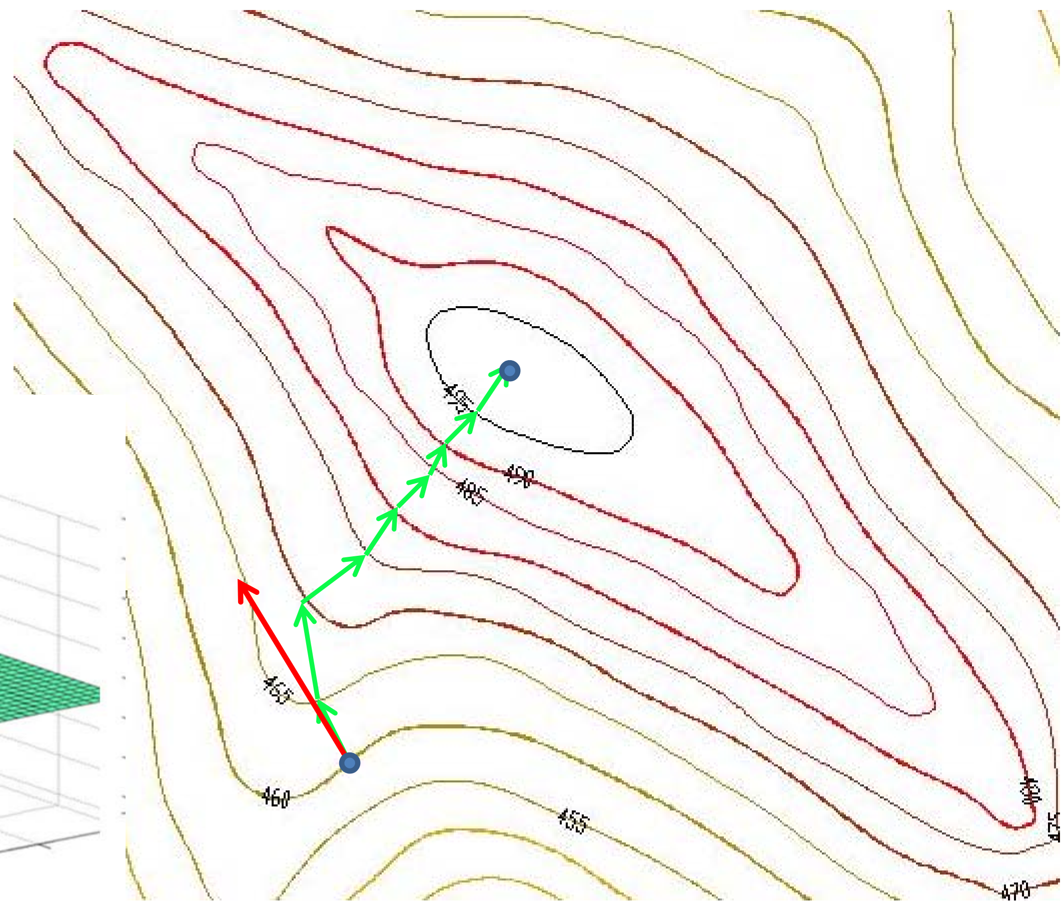
Example 4: Application of the gradient in the optimization problem

Optimization in (x, y) space = finding a maximum of a goal function

$$z = f(x, y)$$

1. Plot contour lines $f(x, y) = const$
2. $\mathrm{grad}\ f$ is normal to contour lines
3. $\mathrm{grad}\ f$ corresponds to the direction of the fastest increase of the goal function
4. Steps along $\mathrm{grad}\ f$ lead to a local maximum



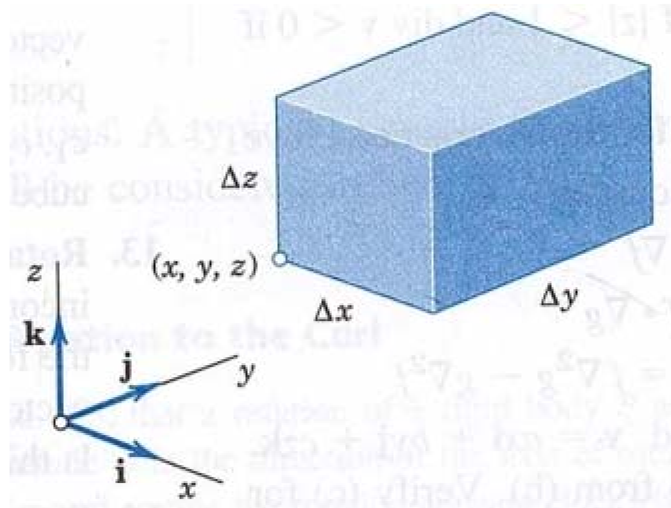


6.8. Gradient, Divergence, and Curl

Divergence

Let's consider a 3D vector field $\mathbf{F}(x, y, z) = F_x(x, y, z)\mathbf{i} + F_y(x, y, z)\mathbf{j} + F_z(x, y, z)\mathbf{k}$. The **divergence** of the vector field is the scalar field div $\mathbf{F}$, which value is equal to

(6.8.5) $$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$



Properties of divergence:

1. Let's consider a small parallelepiped of volume $\Delta V = \Delta x \Delta y \Delta z$. For any face we can define the **flux** of vector field $\mathbf{F}$ as $F = \mathbf{F} \cdot \mathbf{n} \Delta A$, where $\mathbf{n}$ is the unit vector normal to the face and directed outwards and ΔA is the face area. Then the flux through the whole surface is

$$\Delta F(\mathbf{F}) = F_x(x + \Delta x, y, z)\Delta y \Delta z - F_x(x, y, z)\Delta y \Delta z + F_y(x, y + \Delta x, z)\Delta x \Delta z - F_y(x, y, z)\Delta x \Delta z$$
$$+ F_z(x, y, z + \Delta z)\Delta x \Delta y - F_z(x, y, z)\Delta x \Delta y \approx \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)\Delta V = \text{div}\,\mathbf{F}\,\Delta V$$

Now we see that

$$\text{div}\,\mathbf{F} = \lim_{\Delta V \to 0} \frac{\Delta F(\mathbf{F})}{\Delta V} \qquad (6.8.6)$$

Divergence div $\mathbf{F}$ *defines the ratio of the flux of the vector field* $\mathbf{F}$ *through the surface of an (infinitely) small domain to the volume of this domain.*

6.8. Gradient, Divergence, and Curl

2. Divergence is the physical scalar, i.e. its value does not change at any coordinate transformation. Proof is based on Eq. (6.7.6), but also can be performed directly by calculating $\operatorname{div} \mathbf{F}$ after a coordinate transformation.

Example 1: Divergence of a central vector field. The potential energy of a point mass m in the spherically symmetric gravitational field of mass M with center in point (x_0, y_0, z_0) is equal to

$$U(x, y, z) = -GMm/r, \quad r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}.$$

Then

$$\nabla U = -\mathbf{F} = \frac{GMm}{r^2}\left(\frac{x - x_0}{r}\mathbf{i} + \frac{y - y_0}{r}\mathbf{j} + \frac{z - z_0}{r}\mathbf{k}\right) = \frac{GMm}{r^2}\hat{\mathbf{e}}_r$$

$$\frac{\partial(-F_x)}{\partial x} = GMm\frac{\partial}{\partial x}\frac{x - x_0}{r^3} = GMm\frac{r^3 - 3r^2(x - x_0)^2/r}{r^6}$$

$$\nabla \cdot (\nabla U) = -2\frac{GMm}{r^3}$$

Note: Divergence of any central vector field is a central scalar field.

Example 2: Gradient and divergence of a central field in spherical coordinates.

If $(x_0, y_0, z_0) = 0$, then the scalar field $U(x, y, z) = U(r)$ depends only on the distance to the center of coordinates and, thus, it is spherically symmetric with respect to O. Then

$$\nabla U = \frac{\partial U}{\partial r}\hat{\mathbf{e}}_r = \frac{GMm}{r^2}\hat{\mathbf{e}}_r, \qquad \nabla \cdot (\nabla U) = \frac{\partial}{\partial r}\frac{\partial U}{\partial r} = -2\frac{GMm}{r^3}.$$

6.8. Gradient, Divergence, and Curl

The divergence is important for models based on the conservation laws: If $\mathbf{F}$ is a flux density of some physical quantity, then $\text{div}\,\mathbf{F}$ is related density of sources or drains of this quantity.

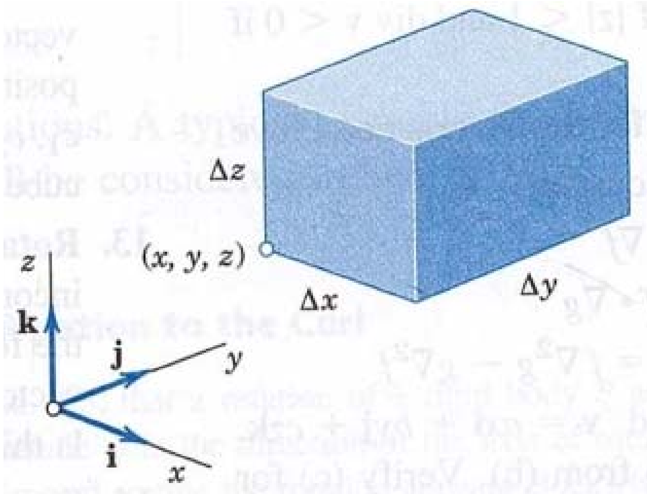
Example 3: Mass conservation in the fluid flow

$\mathbf{F}$ = $\rho\mathbf{v}$, mass flux density of a fluid

$F_x(x, y, z)\Delta y \Delta z = \rho v_x \Delta y \Delta z$, mass flux through the face $\Delta y \Delta z$

$\Delta F(\rho\mathbf{v})$, mass flux through the whole surface

$M = \rho \Delta V$, fluid mass inside the parallelepiped



Mass conservation law (no sources or drains for mass):

$$dM = -\Delta F(\rho\mathbf{v})dt$$

$$\Delta V \frac{\partial \rho}{\partial t} + \Delta F(\rho\mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\Delta F(\rho\mathbf{v})}{\Delta V} = 0$$

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho\mathbf{v}) = 0 : \textbf{Continuity equation}$$

Steady-state flow ($\partial/\partial t = 0$): $\text{div}(\rho\mathbf{v}) = 0$, which means the absence of internal sources or drains of mass inside the fluid flow.

Incompressible flow ($\rho = const$): $\text{div}\,\mathbf{v} = 0$.

Curl

Let's consider a vector field $\mathbf{F}(x, y, z) = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$. The **curl** $\text{curl}\,\mathbf{F}$ of $\mathbf{F}$ is the vector field

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k} \quad (6.7.7)$$

This definition is valid for right-handed Cartesian coordinates. For left-handed coordinates, the definitions includes the negative sign: $\text{curl}\,\mathbf{F} = -\nabla \times \mathbf{F}$.

Properties of curl:

1. The curl is the physical vector, i.e. it does not change at any coordinate transformation between any right-handed coordinates.
2. Let's assume that $\mathbf{v}(\mathbf{r})$ is the velocity field of a rigid body. It is proved in kinematics that the velocity field in the rigid body with respect to pole at point $\mathbf{r}_0$ can be represented in the form

$$\mathbf{v}(\mathbf{r}) = \mathbf{v}(\mathbf{r}_0) + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_0)$$

where $\boldsymbol{\omega}$ is the **angular velocity vector**. Let's use right-handed coordinates where $\boldsymbol{\omega} = \omega\mathbf{k}$. Then $\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_0) = -\omega(y - y_0)\mathbf{i} + \omega(x - x_0)\mathbf{j}$

$$\text{curl}\,\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega(y - y_0) & \omega(x - x_0) & 0 \end{vmatrix} = 2\omega\mathbf{k} = 2\boldsymbol{\omega} \quad \Longrightarrow \quad \boldsymbol{\omega} = \frac{1}{2}\text{curl}\,\mathbf{v} \quad (6.7.8)$$

6.8. Gradient, Divergence, and Curl

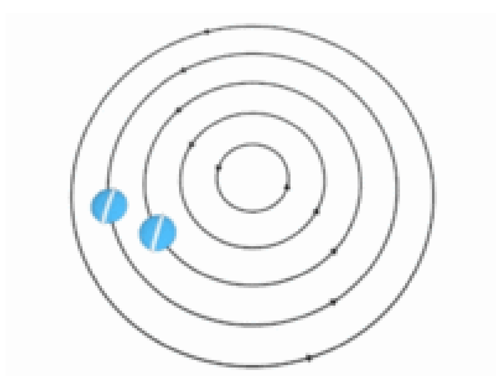
Deformable body: Relative positions of points vary in the course of motion

One can show that for arbitrary deformable body (solid, liquid or gaseous), the kinematic formula can be generalized in the form:

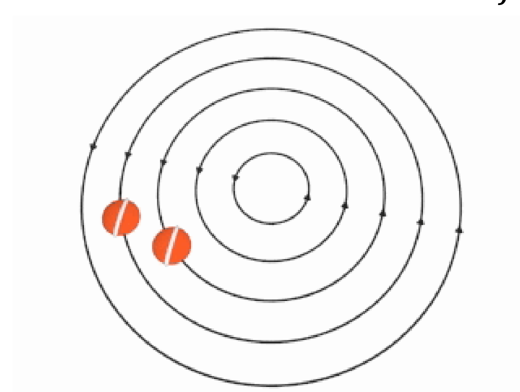
$$\mathbf{v}(\mathbf{r}) = \mathbf{v}(\mathbf{r}_0) + \frac{1}{2}\mathrm{curl}\,\mathbf{v} \times (\mathbf{r} - \mathbf{r}_0) + \mathbf{v}_{def}$$

where $\mathbf{v}_{def}$ is the deformation velocity which describes the change of relative positions of points in the body due to its deformation. Thus, for any body, rigid or deformable, *the curl of the velocity field describes the part of the motion that corresponds to the rigid rotation.*

Rigid body rotation, $\mathbf{v}_{def} = 0$



Deformable body rotation, $\mathbf{v}_{def} \neq 0$



6.8. Gradient, Divergence, and Curl

3. Circulation of the vector field $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j}$ on the plane

$$\Gamma = \int_a^b (F_x dx + F_y dy) = \oint (F_x\mathbf{i} + F_y\mathbf{j}) \cdot \frac{x'\mathbf{i} + y'\mathbf{j}}{\sqrt{(x')^2 + (y')^2}} dt$$

Let's consider a small rectangle in xy-plane that corresponds to increments Δx and Δy . We can introduce the **circulation** $\Delta\Gamma$ of the vector field along the contour of this rectangle:

$$\Delta\Gamma = \mathbf{F}(x,y) \cdot (\Delta x\mathbf{i}) + \mathbf{F}(x+\Delta x, y) \cdot (\Delta y\mathbf{j}) + \mathbf{F}(x, y+\Delta y) \cdot (-\Delta x\mathbf{i}) + \mathbf{F}(x,y) \cdot (-\Delta y\mathbf{j})$$

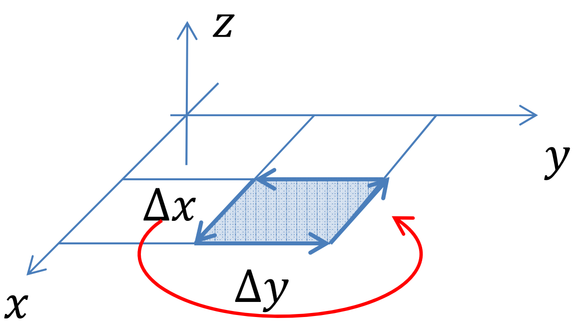
$$= -\frac{F_x(x, y+\Delta y) - F_x(x,y)}{\Delta y} \Delta y \Delta x + \frac{F_y(x+\Delta x, y) - F_y(x,y)}{\Delta x} \Delta x \Delta y$$

$$\approx -\frac{\partial F_x}{\partial y} \Delta y \Delta x + \frac{\partial F_y}{\partial x} \Delta x \Delta y = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \Delta A = (\mathrm{curl}\ \mathbf{F}) \cdot \mathbf{k}\ \Delta A$$

Now we see that

(6.8.9) $$(\mathrm{curl}\ \mathbf{F}) \cdot \mathbf{n} = \lim_{\Delta A \to 0} \frac{\Delta\Gamma}{\Delta A}$$

i.e. *limit of circulation of a vector field over a small contour to the area bounded by this contour is equal to the component of the curl normal to the contour.*



Right hand-side screw rule

6.8. Gradient, Divergence, and Curl

Laplacian

The **Laplacian** of a scalar field $f(x, y, z)$ is

$$\Delta f = \nabla \cdot \nabla f = \nabla^2 f = \mathrm{div}(\mathrm{grad}\, f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} \qquad (6.8.10)$$

Example: The heat conduction equation in the medium with constant properties is

$$\rho c \frac{\partial T}{\partial t} = -\nabla \cdot (-k \nabla T) = k \Delta T$$

Common properties of ∇f, $\nabla \cdot \mathbf{F}$, $\nabla \times \mathbf{F}$, and $\nabla^2 f$:

- ∇f, $\nabla \cdot \mathbf{F}$, $\nabla \times \mathbf{F}$, and $\nabla^2 f$ are invariant with respect to coordinate transformations.
- Linearity, e.g., $\nabla(af + bg) = a\nabla f + b\nabla g$, where a and b are constants.

Special scalar and vector fields:

- Vector field $\mathbf{v}(\mathbf{r})$ is called **gradient (conservative) field**, if there is scalar field $f(\mathbf{r})$, such that $\mathbf{v} = \mathrm{grad}\, f$. Scalar field f is called the **potential** of the vector field $\mathbf{v}$.
- Vector field $\mathbf{v}(\mathbf{r})$ is called **irrotational field** if $\mathrm{curl}\, \mathbf{v} = 0$.
- Vector field $\mathbf{v}(\mathbf{r})$ is called **divergence-free field** if $\mathrm{div}\, \mathbf{v} = 0$.
- Vector field $\mathbf{v}(\mathbf{r})$ is called **central field** if $\mathbf{v} = v(|\mathbf{r}|)\mathbf{r}/|\mathbf{r}|$.

6.8. Gradient, Divergence, and Curl

Properties:

- Any gradient field is irrotational, *i.e.* $\mathrm{curl}\,(\,\mathrm{grad}\,f\,) = 0$.
- Curl produces divergence-free field: $\mathrm{div}\,(\mathrm{curl}\,\mathbf{v}) = 0$.
- Any central field is a gradient field with the potential $f(\mathbf{r}) = u(|\mathbf{r}|), \;\; u(r) = \int v(r)dr + c$.

Example: Irrotational fluid flow.

Fluid flow is called **irrorational** (or **potential**) if it's velocity field is irrorational, i.e. $\mathrm{curl}\,\mathbf{v} = 0$. Now assume that $\mathbf{v} = \nabla\varphi$, then $\mathrm{curl}\,\mathbf{v} = \mathrm{curl}\,(\nabla\varphi) = 0$. φ is called the **velocity potential**.

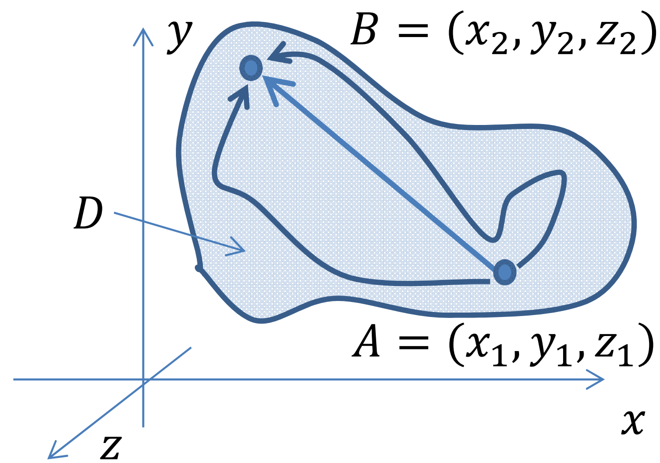
If fluid flow is **incompressible**, when the fluid density is constant, and the velocity field is divergent-free, $\nabla \cdot \mathbf{v} = 0$.

Then the velocity potential in any irrotational incompressible fluid flow should satisfy the **Laplace equation**:

$$\nabla \cdot (\nabla\varphi) = 0 \qquad \text{or} \qquad \Delta\varphi = 0.$$

6.9. Path independent line integrals

A line integral is said to be **path independent** in a domain D if for every pair of endpoints A and B in the domain D, the line integral has the same value for all paths in D that begin in A and end at B.



Theorem 1:

A line integral for the vector field $\mathbf{F}$ with continuous components in D is path independent in D if and only if $\mathbf{F}$ is a gradient field of some potential f, i.e. $\mathbf{F} = \nabla f$.

Proof: We'll prove only the first part of the theorem. Let's assume that $\mathbf{F} = \nabla f$. Then along the integration path f is a function of t: $f(t) = f(x(t), y(t), z(t))$ and

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt = \int_a^b \frac{df}{dt} dt = f(x_2, y_2, z_2) - f(x_1, y_1, z_1) \quad (6.9.1)$$

Thus, the line integral is determined only by values of f in the start and end points, i.e. it is path independent.

Theorem 2:

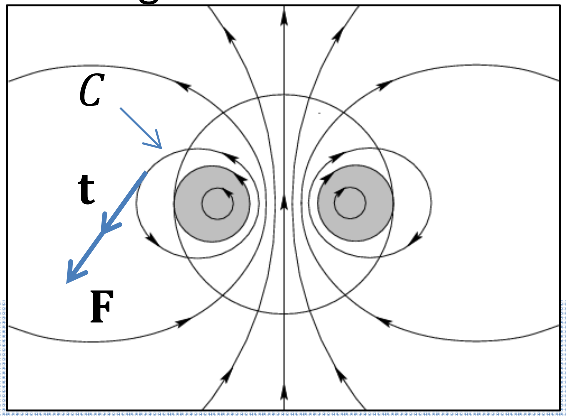
A line integral for the vector field $\mathbf{F}$ with continuous components in D is path independent in D if and only if any circulation of $\mathbf{F}$ (line integral over a closed path in D) is zero.

Proof: From Eq. (6.6.5): $\oint_C \mathbf{F} \cdot d\mathbf{r} = f(x_1, y_1, z_1) - f(x_1, y_1, z_1) = 0$

6.9. Path independent line integrals

Consequence 1: If field lines are closed curves in some region, the vector field is non-conservative (non-gradient) and line integrals are path dependent in this region.

Proof: If the field lines are closed, chose a path of integration which coincides with a close field line. Then in any point of such line $\mathbf{F} \cdot d\mathbf{r} = \mathbf{F} \cdot \mathbf{t} dt > 0$ and, circulation can not be equal to zero.



Consequence 2: Circulation of any gradient field is zero.

Theorem 3:
A line integral for the vector field $\mathbf{F}$ with continuous components in D is path independent in D if and only if the differential form

$$\mathbf{F} \cdot d\mathbf{r} = F_x dx + F_y dy + F_y dz \qquad (6.9.2)$$

is the total differential of some function f.

Proof: According to theorem 1, the line integral is path independent if and only if $\mathbf{F} = \nabla f$. But then $\mathbf{F} \cdot d\mathbf{r} = df$, i.e. the differential form (4.6.6) is the total differential of the field potential f.

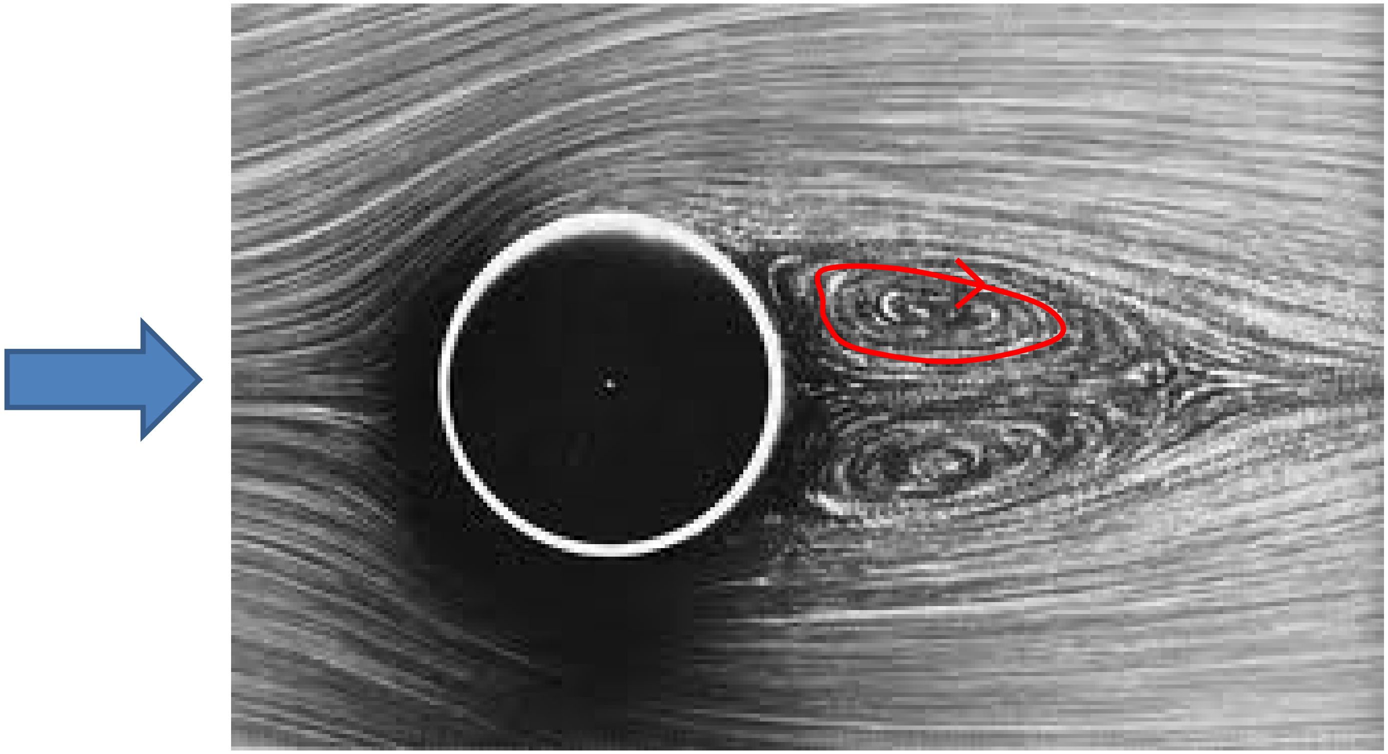
From the theorems 1-3 we see that **the following statements are equivalent**:

1. Line integral for $\mathbf{F}$ is path independent.
2. $\mathbf{F}$ is a gradient field.
3. Any circulation of $\mathbf{F}$ is zero.
4. Differential form (4.6.6) for $\mathbf{F}$ is the total differential (Criterion for exactness will be derived from Stokes's theorem later).

6.9. Path independent line integrals

Example: Circular cylinder in cross flow with attached circulation zone (moderate Reynolds numbers)

Steady-state flow: streamlines = trajectories of fluid particles



- The circulation zone appears due to action of fluid viscosity which induces the flow detachment.
- For the fluid velocity field, the line integral is path dependent.
- The fluid velocity field is non-potential, the theory of potential flows can not be applied.

6.9. Path independent line integrals

Path independent integrals of work in mechanics: Existence of potential energy

Let's consider a point mass that is places into a force field $\mathbf{F}(\mathbf{r})$, i.e. in every point with position vector $\mathbf{r}$, the force $\mathbf{F}(\mathbf{r})$ is exerted on the point mass.

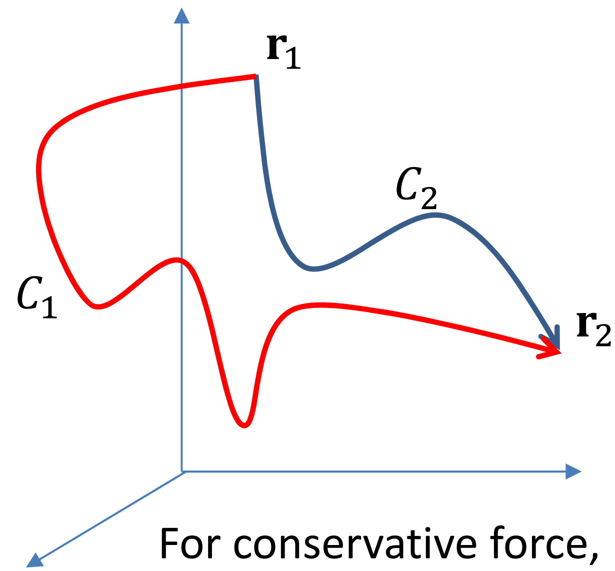
The work W done by this force along the trajectory C of this point mass started in point $\mathbf{r}_1$ and ended in point $\mathbf{r}_2$ is given by the line integral

$$W(C) = \int_C \mathbf{F} \cdot d\mathbf{r} \qquad (6.9.3)$$

The force (force field) $\mathbf{F}(\mathbf{r})$ is called **conservative** if line integral in Eq. (6.9.1) is path independent, otherwise $\mathbf{F}(\mathbf{r})$ is called **non-conservative** force.

According to theorems 1-3, we can give the following alternative definitions of the conservative force: $\mathbf{F}(\mathbf{r})$ is conservative if

1. Line integral for $\mathbf{F}$ is path independent, i.e. work of force does not depend on the shape of trajectory, but is determined only by its start $\mathbf{r}_1$ and end $\mathbf{r}_2$ points.
2. $\mathbf{F}$ is a gradient field: $\mathbf{F} = -\nabla U$, where scalar field $U(\mathbf{r})$ is called the **potential energy**.
3. Any circulation of $\mathbf{F}$ is zero, i.e. work W done by $\mathbf{F}$ at any closed trajectory ($\mathbf{r}_1 = \mathbf{r}_2$) is equal to zero.
4. Differential form $\mathbf{F} \cdot d\mathbf{r}$ is the total differential.



For conservative force, $W(C_1) = W(C_2)$

6.9. Path independent line integrals

Thus, conservative nature of the force is equivalent to the existence of the potential energy $U(\mathbf{r})$. Then according to Eq. (6.9.1)

$$W(C) = W(\mathbf{r}_1, \mathbf{r}_2) = \int_C \mathbf{F} \cdot d\mathbf{r} = U(\mathbf{r}_2) - U(\mathbf{r}_2) \quad (6.9.4)$$

This equation together with Newton's second law of motion results in the conservation law of the total mechanical energy.

Components of the potential force depend only on coordinates (position of the point mass) and do not depend, e.g., on the velocity vector, acceleration vector, time, etc.

Conservative force fields $\mathbf{F}(\mathbf{r})$ play very important role in mechanics. In particular, all "fundamental" interaction forces including

- Gravity force
- Electrostatic force
- Interaction forces between individual atoms

are all conservative force.

At the same time, friction force are non-conservative, since it usually depends on velocity.

In order to define a conservative forces field, it is sufficient to defined the potential energy of this field as a function of coordinates, $U = U(x, y, z)$. This approach is widely used, e.g., for formulation of interatomic forces in **Molecular Dynamics** (MD) simulations.

6.10. Double integral

"Ordinary" integral = integral of a function of a single argument over length in 1D domain.
Double integral = integral of a function of two arguments over area in 2D domain.
Triple integral = integral of a function of three arguments over volume in 3D domain.

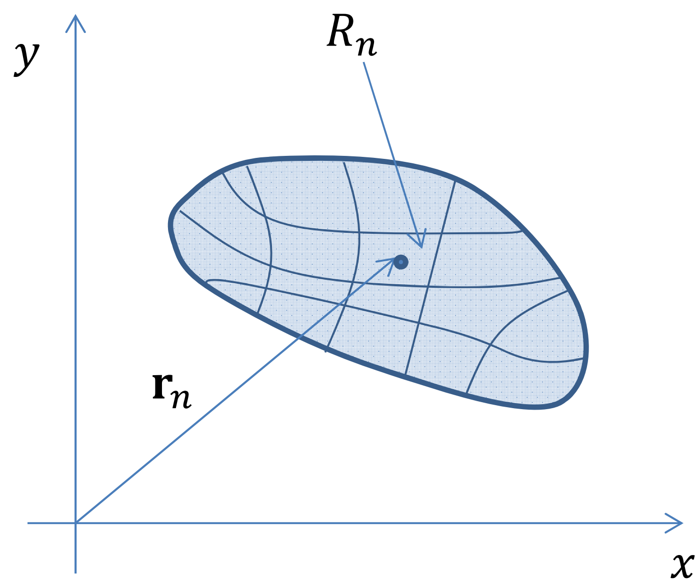
Let's consider a two-dimensional space – xy-plane, a region R - set of points and on this plane, and a 2D scalar field $f(\mathbf{r}) = f(x, y)$ that is defined in every point of R.

Let's do the following steps:

1. Divide domain R into N subdomains R_n ($n = 1, .. N$, $N >> 1$) of area ΔA_n.
2. Inside every subdomain choose a point $\mathbf{r}_n$ and value of the scalar field $f(\mathbf{r}_n)$.
3. Calculate $\Delta A_{max} = \max(\Delta A_n)$.
4. Calculate the sum $\sum_{n=1}^{N} f(\mathbf{r}_n)\Delta A_n$.

Then the **double integral** over domain R is the limit

(6.10.1) $$\iint_R f(\mathbf{r})dA = \lim_{\Delta A_{max}\to 0} \sum_{n=1}^{N} f(\mathbf{r}_n)\Delta A_n$$

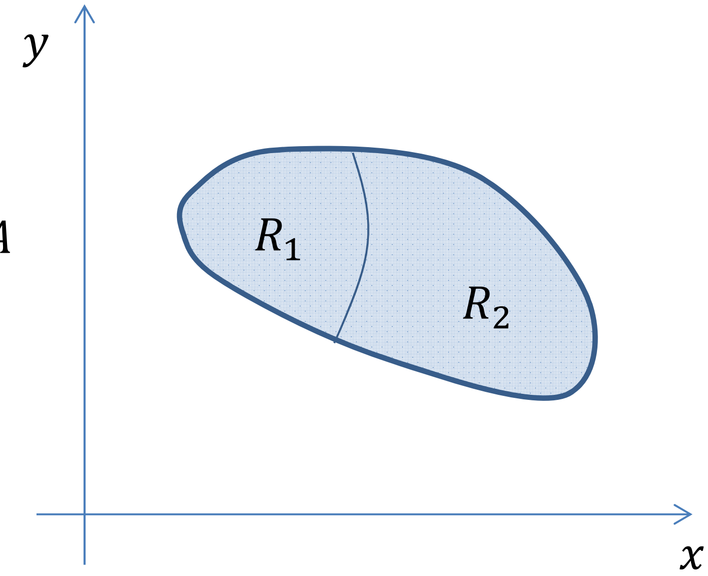


In Cartesian coordinates $dA = dxdy$ and

(6.10.2) $$\iint_R f(\mathbf{r})dA = \iint_R f(x, y)dxdy$$

6.10. Double integral

Basic properties of the double integral:



1. Linearity ($a, b = const$)

$$\iint_R [af(\mathbf{r}) + bg(\mathbf{r})]dA = a\iint_R f(\mathbf{r})dA + b\iint_R g(\mathbf{r})dA$$

1. Additivity

$$\iint_R f(\mathbf{r})dA = \iint_{R_1} f(\mathbf{r})dA + \iint_{R_2} f(\mathbf{r})dA$$

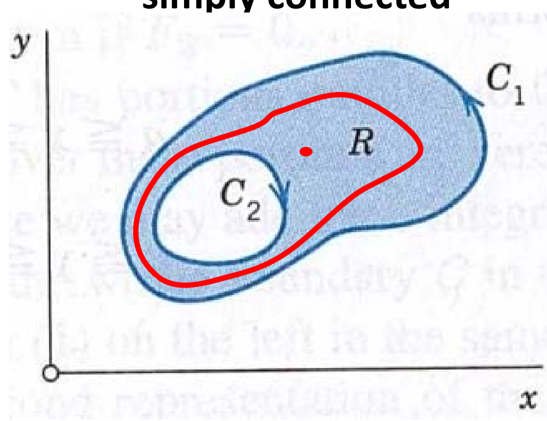
3. **Mean value theorem**: If R is simply connected and $f(\mathbf{r})$ is continuous in R, then there is such point $\mathbf{r}_0$ in R that

$$\iint_R f(\mathbf{r})dA = f(\mathbf{r}_0)A$$

where A is the area of R. (R is called **simply connected**, if any closed curve in R can be continuously shrunk to any point in R without leaving R)

4. Physical meaning of the double integral: If $f = 1$, then the double integral is the area of R.

Example of R which is not simply connected



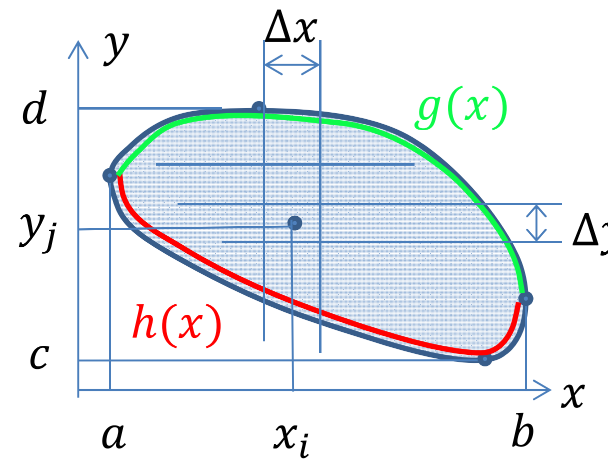
6.10. Double integral

Calculation of the double integral by two successive integrations for a simple domain in Cartesian coordinates

We say that R is **simple with respect to x** if the boundary of R can be described by two boundary functions $h(x)$ and $g(x)$ as shown in the figure.
Let's introduce subdivision of R with the help of rectangular cells of even size $\Delta A_n = \Delta x \Delta y$:

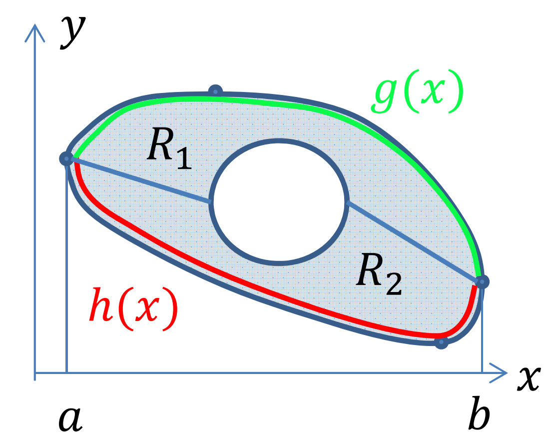
$$\iint\limits_R f(x,y)dxdy = \lim_{\Delta x \Delta y \to 0} \sum_{i=1}^{N} \sum_{j=1}^{M} f(x_i, y_j) \Delta x \Delta y =$$

(6.10.3)
$$\lim_{\Delta x \Delta y \to 0} \sum_{i=1}^{N} \left(\sum_{j=1}^{M} f(x_i, y_j) \Delta y \right) \Delta x = \int_a^b \left(\int_{h(x)}^{g(x)} f(x,y)dy \right) dx$$



If R is simple with respect to y, then

(6.10.4)
$$\iint\limits_R f(x,y)dxdy = \int_c^d \left(\int_{p(y)}^{q(y)} f(x,y)dx \right) dy$$



If R is not simple, it usually can be represented as a set of simple adjoin subdomains, so the double integral can be calculated using (6.10.2) and (6.10.3) and additivity.

6.10. Double integral

Note 1: If domain R is a rectangle $a \le x \le b$, $c \le y \le d$ then

$$\iint_R f(x,y)dxdy = \int_a^b \left(\int_c^d f(x,y)dy \right) dx = \int_c^d \left(\int_a^b f(x,y)dx \right) dy$$

Note 2: If domain R is a rectangle $a \le x \le b$, $c \le y \le d$ and $f(x,y) = f_1(x)\, f_2(y)$, then

$$\iint_R f(x,y)dxdy = \left(\int_a^b f_1(x)dx \right) \left(\int_c^d f_2(y)dy \right)$$

Example: Mass, moment of inertia with respect to z-axis, and internal energy of a cylindrical body of length L

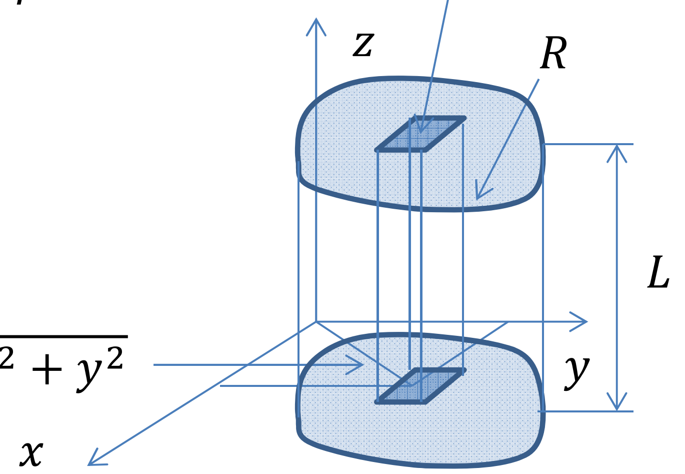
$$dV = Ldxdy$$

$$M = L \iint_R \rho(x,y)dxdy \qquad dM = \rho dV$$

$$I_z = L \iint_R \rho(x,y) r^2 dxdydz = L \iint_R \rho(x,y)(x^2+y^2)dxdy$$

$$U = L \iint_R \rho(x,y) u(x,y,z)dxdy$$

$$r = \sqrt{x^2+y^2}$$



Change of variables in the double integral

Let's introduce new variables $u = u(x, y)$ and $v = v(x, y)$. We assume that this transformation of coordinates is **non-degenerate** in R, so $x = x(u, v)$ and $y = y(u, v)$ exist. In new coordinates, domain R transforms into domain R_*.

Question: How should we change the integrand in order to guarantee that the double integral in old and new coordinates has the same value, $I_{xy} = I_{uv}$?

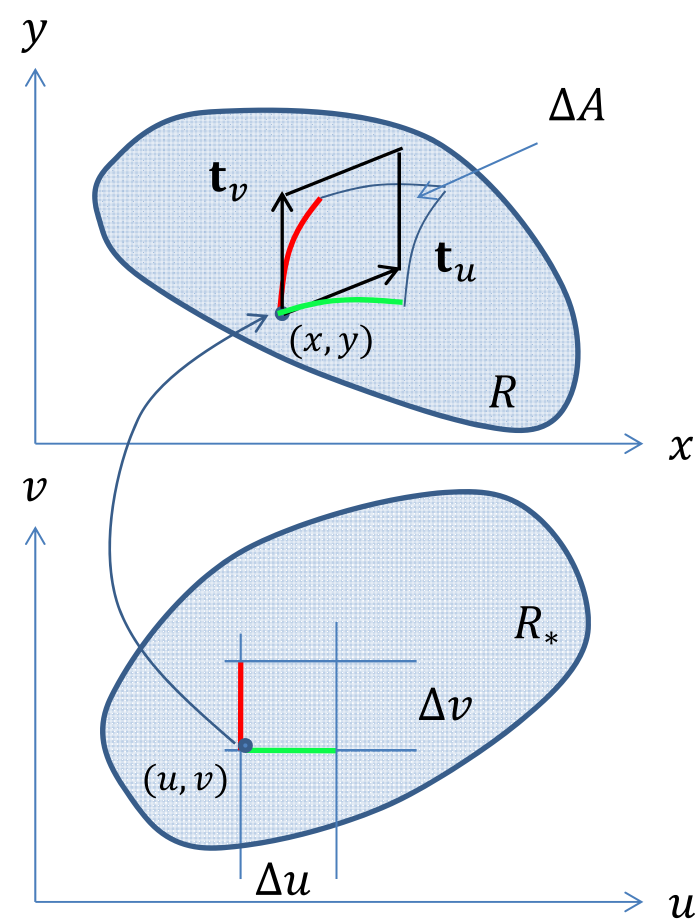
$$I_{xy} = \iint_R f(x, y) dx dy$$

$$I_{uv} = \iint_{R_*} \tilde{f}(u, v) du dv$$

Obviously $\tilde{f}(u(x, y), v(x, y)) \neq f(x, y)$, e.g.
if $f = 1, u = ax, v = by$, then $\tilde{f} = 1/(ab)$.

Let's introduce a rectangular mesh of cells on the uv-plane. Then

$$\iint_{R_*} \tilde{f}(u, v) du dv = \lim_{\Delta u \Delta v \to 0} \sum_{i=1}^{N} \sum_{j}^{M} \tilde{f}(u_i, v_j) \Delta u \Delta v$$



For $I_{xy} = I_{uv}$:

$$\tilde{f}(u, v) \Delta u \Delta v = f(x(u, v), y(u, v)) \Delta A$$

6.10. Double integral

$$\tilde{f}(u,v)\Delta u\Delta v = f(x(u,v),y(u,v))\Delta A$$

We should calculate ΔA. Every rectangular cell of area $\Delta u\Delta v$ in uv-plane corresponds to a curvilinear cell of area ΔA in xy-plane. If Δu and Δv are sufficiently small then the cell in xy-plane is close to a parallelogram with edges parallel to vectors

$$\mathbf{t}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}, \qquad \mathbf{t}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}$$

The area of the parallelogram is

$$\Delta A = |(\mathbf{t}_u\Delta u)\times(\mathbf{t}_v\Delta v)| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta u & 0 \\ \frac{\partial x}{\partial v}\Delta v & \frac{\partial y}{\partial v}\Delta v & 0 \end{matrix} \right\| = |J|\Delta u\Delta v$$

where J is the **Jacobian determinant**, determinant of the 2D Jacobian matrix,

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

And the rule of the variable change in the double integral is

(6.10.5) $$\iint\limits_R f(x,y)dxdy = \iint\limits_{R_*} f(x(u,v),y(u,v))|J|dudv$$

6.10. Double integral

If we want to change variables from the old variables x, y to the new variables : $u = u(x, y)$ and $v = v(x, y)$, we must:

1. Find an image R_* of the domain R on the uv-plane.
2. Find the inverse transform $x = x(u, v)$ and $y = y(u, v)$.
3. Calculate the Jacobian of the inverse transform.
4. Change the integrand according to Eq. (6.10.5).

Note: Usually the purpose of the variable change in the double integral is to make the shape of boundaries of R_* as geometrically simple as possible.

Example: Calculation of the area of the circle.

$$A = \iint_R dxdy\,, \text{where } R\colon\ x^2 + y^2 \leq (D/2)^2$$

Let's perform a transform to polar coordinates: $x = r\cos\theta\,, y = r\sin\theta, J = r$.

In polar coordinates, the image R_* is the rectangle: $0 \leq r \leq D/2$, $0 \leq \theta \leq 2\pi$

$$A = \iint_R dxdy = \int_0^{2\pi} \left(\int_0^{D/2} rdr \right) d\theta = \pi \left(\frac{D}{2}\right)^2$$

6.11. Triple integral

"Ordinary" integral = integral of a function of a single argument over length in 1D domain.
Double integral = integral of a function of two arguments over area in 2D domain.
Triple integral = integral of a function of three arguments over volume in 3D domain.

Assume that we consider a three-dimensional xyz-space, a region T - set of points and in this scape, and a 3D scalar field $f(\mathbf{r}) = f(x, y, z)$ that is defined in every point of T.

Let's do the following steps

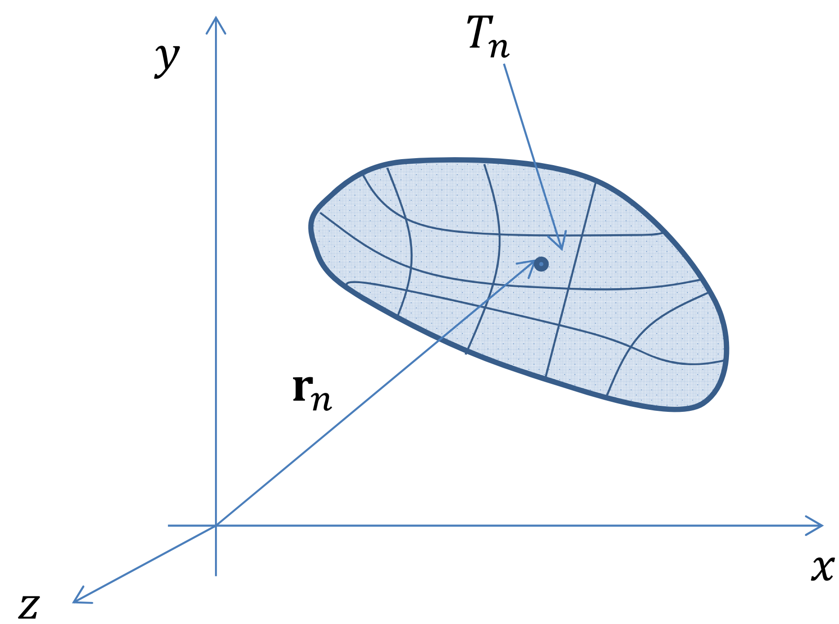
1. Divide domain T into N subdomains T_n $(n = 1, .. N, N >> 1)$ of volume ΔV_n.
2. Inside every subdomain choose a point $\mathbf{r}_n$ and value of the scalar field $f(\mathbf{r}_n)$.
3. Calculate $\Delta V_{max} = \max(\Delta V_n)$.
4. Calculate the sum $\sum_{n=1}^{N} f(\mathbf{r}_n)\Delta V_n$.

Then the **triple integral** over domain T is the limit

(6.11.1) $$\iiint_T f(\mathbf{r})dV = \lim_{\Delta V_{max} \to 0} \sum_{n=1}^{N} f(\mathbf{r}_n)\Delta V_n$$

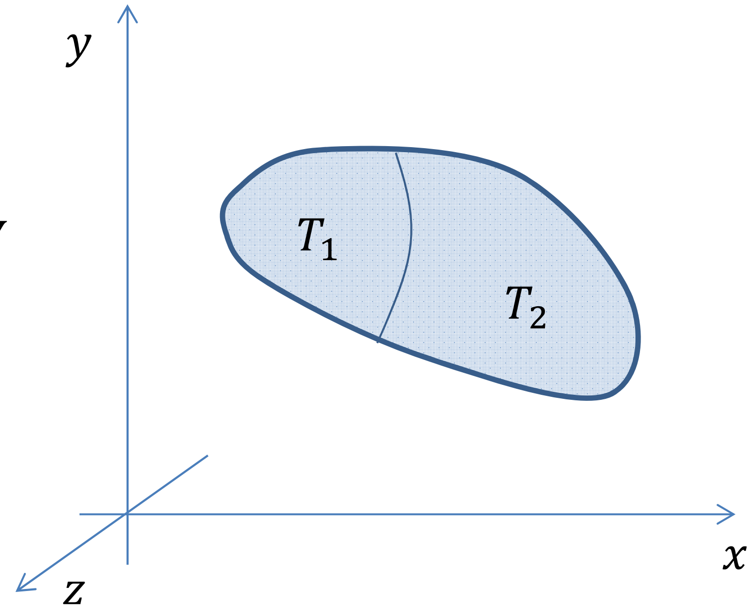
In Cartesian coordinates $dV = dxdydz$ and

(6.11.2) $$\iiint_T f(\mathbf{r})dV = \iiint_T f(x, y, z)dxdydz$$



6.11. Triple integral

Basic properties of the triple integral:



1. Linearity ($a, b = const$)

$$\iiint_T [af(\mathbf{r}) + bg(\mathbf{r})]dV = a\iiint_T f(\mathbf{r})dV + b\iiint_T g(\mathbf{r})dV$$

1. Additivity

$$\iiint_T f(\mathbf{r})dV = \iiint_{T_1} f(\mathbf{r})dV + \iiint_{T_2} f(\mathbf{r})dV$$

3. **Mean value theorem**: If T is simply connected and $f(\mathbf{r})$ is continuous in T, then there is such point $\mathbf{r}_0$ in T that

$$\iiint_T f(\mathbf{r})dV = f(\mathbf{r}_0)V$$

where V is the volume of T. (T is called simply connected, if any closed curve in T can be continuously shrunk to any point in T without leaving T)

4. Physical meaning of the triple integral: If $f = 1$, then triple integral is the volume of T.

6.11. Triple integral

Calculation of the double integral by three successive integrations for a simple domain in Cartesian coordinates

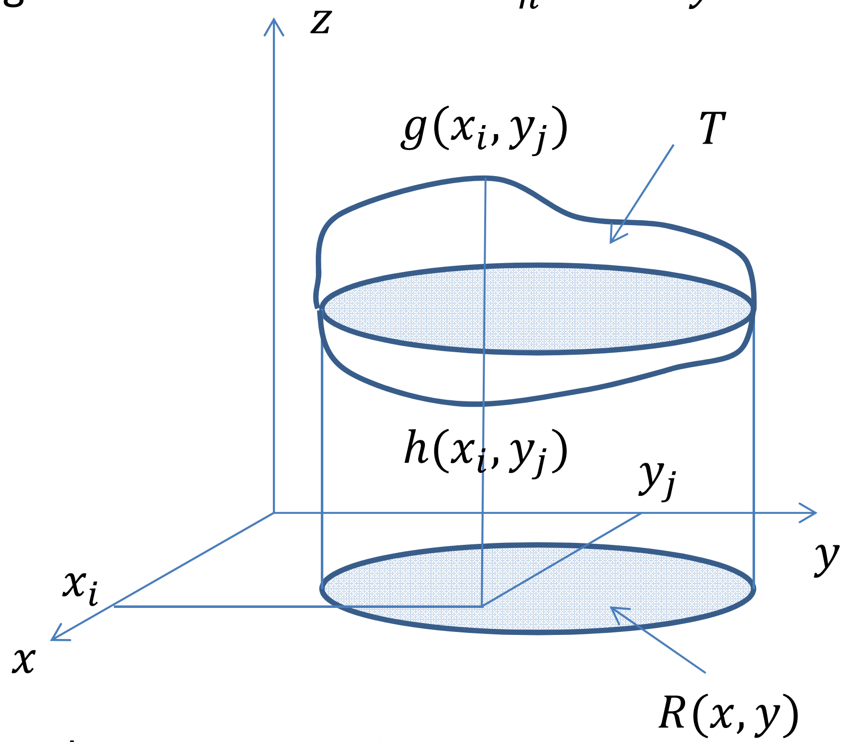
Domain T is called **simple with respect to some coordinate**, e.g., z, if the surface of the domain can be described by two boundary functions $z = g(x, y)$ and $z = h(x, y)$, see figure. For the domain simple with respect to z, the triple integral can be calculated by successive integration. Corresponding formulae can be obtained similar to the case of the double integral by introducing subdivision of T with the help of “rectangular” cells of even size $\Delta V_n = \Delta x \Delta y \Delta z$:

$$\iiint\limits_T f(x,y,z)dxdydz =$$

$$\lim_{\Delta x \Delta y \Delta z \to 0} \sum_{i=1}^{N} \sum_{j=1}^{M} \sum_{k=1}^{L} f(x_i, y_j, z_k) \Delta x \Delta y \Delta z =$$

$$\lim_{\Delta x \Delta y \to 0} \sum_{i=1}^{N} \sum_{j=1}^{M} \left(\sum_{k=1}^{L} f(x_i, y_j, z_k) \Delta z \right) \Delta x \Delta y =$$

(6.11.3) $$\iint\limits_{R(x,y)} \left(\int\limits_{h(x,y)}^{g(x,y)} f(x,y,z)dz \right) dxdy$$



The double integral can be further calculated with two successive integrations if $R(x, y)$ is simple with respect to either x or y.

6.11. Triple integral

(6.11.4) $$\iiint_T f(x,y,z)dxdydz = \int_a^b \left(\int_{h(x)}^{g(x)} \left(\int_{h(x,y)}^{g(x,y)} f(x,y,z)dz \right) dy \right) dx$$

Note 1: If domain T is the bar $a \le x \le b,\ c \le y \le d,\ g \le z \le h$ then

$$\iiint_T f(x,y,z)dxdydz = \int_a^b \left(\int_c^d \left[\int_g^h f(x,y,z)dz \right] dy \right) dx$$

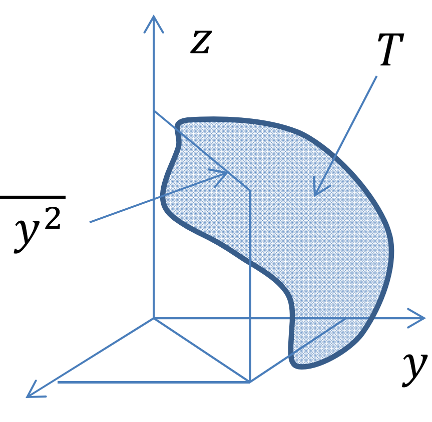
Note 2: If domain T is the bar $a \le x \le b$, $c \le y \le d$, $g \le z \le h$ and $f(x,y,z) = f_1(x)\, f_2(y) f_3(z)$, then

$$\iiint_T f(x,y,z)dxdydz = \left(\int_a^b f_1(x)dx \right) \left(\int_c^d f_2(y)dy \right) \left(\int_g^h f_3(z)dz \right)$$

Example: Mass and moment of inertia of a 3D body with respect to z-axis

$$M = \iiint_T \rho(x,y,z)dxdydz$$

$$r = \sqrt{x^2 + y^2}$$

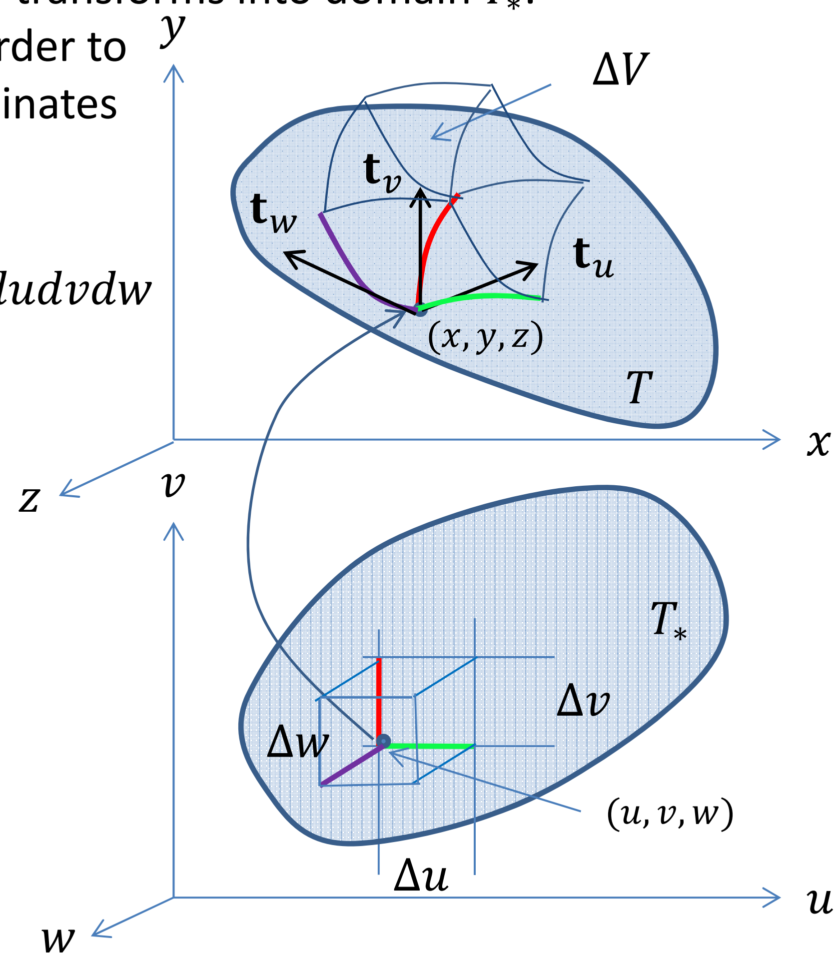


$$I_z = \iiint_T \rho(x,y,z) r^2 dxdydz = \iiint_T \rho(x,y,z)(x^2 + y^2)dxdydz$$

Change of variables in the triple integral

Let's introduce new variables $u = u(x,y,z)$, $v = v(x,y,z)$, and $w = w(x,y,z)$. We assume that this transform of coordinate is non-degenerate in T, so $x = x(u,v,w)$, $y = y(u,v,w)$, and $z = z(u,v,w)$ exist. In new coordinates, the domain T transforms into domain T_*.

Question: How should we change the integrand f in order to guarantee that the triple integral in old and new coordinates has the same value, $I_{xyz} = I_{uvw}$?



$$I_{xyz} = \iiint_T f(x,y,z)dxdydz\,, I_{uvw} = \iiint_{T_*} \tilde{f}(u,v,w)dudvdw$$

Let's introduce a rectangular mesh of cells in the uvw-space. Then

$$\iiint_{T_*} \tilde{f}(u,v,w)dudvdw$$

$$= \lim_{\Delta u \Delta v \Delta w \to 0} \sum_{i=1}^{N} \sum_{j=1}^{M} \sum_{k=1}^{L} \tilde{f}(u_i, v_j, w_k)\Delta u \Delta v \Delta w$$

For $I_{xyz} = I_{uvw}$: $\tilde{f}(u,v,w)\Delta u \Delta v \Delta w = f(x(u,v,w), y(u,v,w), z(u,v,w))\Delta V$

6.11. Triple integral

$$\tilde{f}(u,v,w)\Delta u\Delta v\Delta w = f(x(u,v,w),y(u,v,w),z(u,v,w))\Delta V$$

We should calculate ΔV. Every rectangular cell of volume $\Delta u\Delta v\Delta w$ in uvw-space corresponds to a curvilinear cell of volume ΔV in xyz-space. If Δu, Δv, and Δw are sufficiently small then the cell in xyz-space is close to a parallelepiped with edges parallel to vectors

$$\mathbf{t}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}, \qquad \mathbf{t}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}, \qquad \mathbf{t}_w = \frac{\partial x}{\partial w}\mathbf{i} + \frac{\partial y}{\partial w}\mathbf{j} + \frac{\partial z}{\partial w}\mathbf{k}$$

The volume of the parallelepiped is

$$\Delta V = |(\mathbf{t}_u\Delta u)\cdot[(\mathbf{t}_v\Delta v)\times(\mathbf{t}_w\Delta w)]| = \left\| \begin{matrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta u & \frac{\partial z}{\partial u}\Delta u \\ \frac{\partial x}{\partial v}\Delta v & \frac{\partial y}{\partial v}\Delta v & \frac{\partial z}{\partial v}\Delta v \\ \frac{\partial x}{\partial w}\Delta w & \frac{\partial y}{\partial w}\Delta w & \frac{\partial z}{\partial w}\Delta w \end{matrix} \right\| = |J|\Delta u\Delta v\Delta w$$

where J is the **Jacobian determinant**, the determinant of the 3D Jacobian matrix,

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

6.11. Triple integral

And the rule of the variable change in the triple integral is

(6.11.5) $$\iiint_T f(x,y,z)dxdydz = \iiint_{T_*} f(x(u,v,w), y(u,v,w), z(u,v,w))|J|dudvdw$$

If we want to change variables from the old variables x, y, z to the new variables $u = u(x,y,z)$, $v = v(x,y)$, $w = w(x,y,z)$ we must:

1. Find an image T_* of the domain T in the uvw-space.
2. Find the inverse transform $x = x(u,v,w)$, $y = y(u,v,w)$, and $z = z(u,v,w)$.
3. Calculate the Jacobian determinant of the inverse transform.
4. Change the integrand according to Eq. (6.11.5).

Note: Usually the purpose of the variable change in the triple integral is to make the shape of boundaries of T_* as geometrically simple as possible.

Example: Calculation of the volume of a sphere.

$$V = \iiint_T dxdydz \text{, where } T\text{: } x^2 + y^2 + z^2 \leq (D/2)^2$$

Transform to spher. coordinates: $x = r\cos\theta$, $y = r\sin\theta\cos\varphi$, $z = r\sin\theta\sin\varphi$, $J = r^2\sin\theta$.
In spherical coordinates, image T_* is the bar: $0 \leq r \leq D/2$, $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$

$$V = \iiint_T dxdydz = \int_0^{2\pi}\left[\int_0^{\pi}\left(\int_0^{D/2} r^2 dr\right)\sin\theta\, d\theta\right]d\varphi = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3$$

6.12. Surface integrals of scalar and vector fields

Let's consider some surface S given by the parametric representation:

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

where (v,u) vary over a region R on the uv-plane.
Let's assume that S is piecewise smooth, so that S has a normal vector $\mathbf{N} = \mathbf{t}_u \times \mathbf{t}_v$ and a unit normal vector $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$ at every point except perhaps some edges or cusps. In every point, where $\mathbf{N}$ exists, $dA = |\mathbf{N}|dudv$.

For a given scalar field $f(x,y,z)$ and surface S, the **surface integral of the scalar field** is defined as

$$\iint_S f dA \equiv \iint_R f\big(\mathbf{r}(u,v)\big)|\mathbf{N}|dudv \qquad (6.12.1)$$

Note: Since the RHS in Eq. (6.12.1) depends only on $|\mathbf{N}|$, Eq. (6.12.1) provides a unique value independently on the surface orientation and, in particular, is valid for a nonorientable surface.

For a given vector field $\mathbf{F}(x,y,z)$ and oriented surface S, the **surface integral of the vector field** is defined as

$$\iint_S \mathbf{F}\cdot\mathbf{n} dA \equiv \iint_R \mathbf{F}\big(\mathbf{r}(u,v)\big)\cdot\mathbf{N}(u,v)dudv \qquad (6.12.2)$$

Note: Since RHS in Eq. (6.12.2) depends on $\mathbf{N}$, surface integral (6.12.2) changes its sign when the orientation of the surface changes.

6.12. Surface integrals of scalar and vector fields

Properties of the surface integrals:

1. If we introduce direction cosines for the surface normal such that $\mathbf{n} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$, then $\cos\alpha\,dA = dydz$, $\cos\beta\,dA = dxdz$, and $\cos\gamma\,dA = dxdy$, so that

$$\mathbf{F}\cdot\mathbf{n}\,dA = F_x dydz + F_y dxdz + F_z dxdy$$

   and the surface integral can be calculated as

$$\iint_S \mathbf{F}\cdot\mathbf{n}dA = \iint_S (F_x dydz + F_y dxdz + F_z dxdy)$$

2. Linearity

$$\iint_S (c_1\mathbf{F}_1 + c_2\mathbf{F}_2)\cdot\mathbf{n}dA = c_1\iint_S \mathbf{F}_1\cdot\mathbf{n}dA + c_2\iint_S \mathbf{F}_2\cdot\mathbf{n}dA$$

3. Additivity

$$\iint_S \mathbf{F}\cdot\mathbf{n}dA = \iint_{S_1} \mathbf{F}\cdot\mathbf{n}dA + \iint_{S_2} \mathbf{F}\cdot\mathbf{n}dA$$

4. Surface integral of the scalar field $f = 1$ gives the area of the surface S

$$A = \iint_S dA = \iint_R |\mathbf{N}|dudv = \iint_R |\mathbf{t}_u \times \mathbf{t}_v|dudv$$

6.12. Surface integrals of scalar and vector fields

5. Definition of the surface integral in terms of the sum limit.
   Let's consider a smooth surface S given by the parametric representation $\mathbf{r} = \mathbf{r}(u, v)$ and let's divide it (e.g., by grid lines $u = const$ and $v = const$) into a large number N of subsurfaces ΔS_i. For every subsurface, let's define a point $\mathbf{r}_i$, which belongs to the subsurface, $\mathbf{n}_i$ is the unit normal to the surface in point $\mathbf{r}_i$, ΔA_i is the area of the subsurface, and $\Delta A_{max} = \max(A_i)$. Then

$$\sum_{i=1}^{N} \mathbf{F}(\mathbf{r}_i) \cdot \mathbf{n}_i \, \Delta A_i \xrightarrow[(N\to\infty)]{\Delta A \to 0} \int_S \mathbf{F} \cdot \mathbf{n} \, dA$$

Example 1: Flux density and flux.

In applications, the vector field $\mathbf{F}$ often has the meaning of the **flux density** of some physical quantity, i.e. amount of the physical quantity transferred through the surface of unit area perpendicular to the field vector per unit time.

$\mathbf{F} = \rho \mathbf{v}$: Mass flux density in the fluid flow.

$\mathbf{F} = \mathbf{q}$: Heat flux density.

Then the surface integral of a such vector field has meaning of the **flux**, i.e. the amount of the physical quantity transferred through a whole surface S:

$$\int_S \rho \mathbf{v} \cdot \mathbf{n} \, dA \ : \text{Mass flux}; \qquad \int_S \mathbf{q} \cdot \mathbf{n} \, dA \ : \text{Heat flux}$$

6.12. Surface integrals of scalar and vector fields

Example 2: Moment of inertia of a surface about z-axis.

In order to calculate the moment of inertia of a surface (thin shell) with respect to z-axis, we can introduce the scalar field $f(\mathbf{r}) = \sigma(\mathbf{r})(x^2 + y^2)$, where $\sigma(\mathbf{r})$ is the surface mass density (mass of the thin shell per unit area of the shell). Then

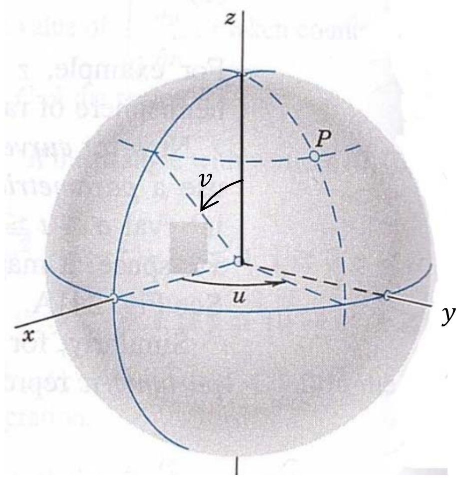
$$I_z = \iint_S \sigma(\mathbf{r})(x^2 + y^2)dA = \iint_R \sigma\big(\mathbf{r}(u,v)\big)(x^2(u,v) + y^2(u,v))|\mathbf{N}|dudv$$

Example 2a: Moment of inertia of a homogeneous spherical shell ($\sigma(\mathbf{r}) = \sigma_0 = const$) with respect to an axis going through the sphere center.

Parametric representation of the sphere:

$$\mathbf{r} = R\,(\sin v \cos u\,\mathbf{i} + \sin v \sin u\ \mathbf{j} + \cos v\,\mathbf{k})$$
$$x(u,v) = R \sin v \cos u$$
$$y(u,v) = R \sin v \sin u$$
$$dA = |\mathbf{N}|dudv = R^2 \sin v\, dudv$$

$$0 \le u \le 2\pi$$
$$0 \le v \le \pi$$



$$I_z = \iint_S \sigma(\mathbf{r})(x^2 + y^2)dA = \sigma_0 \int_0^{\pi}\int_0^{2\pi} R^4 \sin^3 v du\, dv =$$

$$\sigma_0 R^4 (2\pi)\left(-\cos v + \frac{\cos^3 v}{3}\right)\Bigg|_{v=0}^{v=\pi} = \frac{2}{3}(4\pi\sigma_0\ R^2)R^2 = \frac{2}{3} mR^2$$

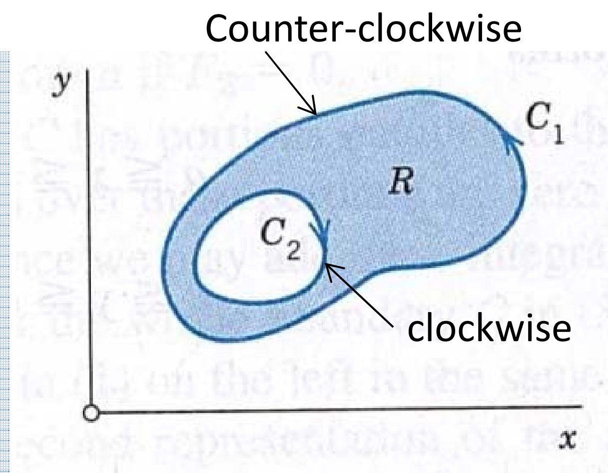
6.13. Green's theorem for a plane

Green's theorem:

Let R be a closed bounded region in the xy-plane whose boundary C consists of piecewise smooth curves. For every closed path C_i we choose direction such that R is on the left as we advance in the direction of integration.

Let $P(x,y)$ and $Q(x,y)$ be functions that are continuous and have continuous derivatives $\partial P/\partial y$ and $\partial Q/\partial x$ everywhere in some domain containing R. Then

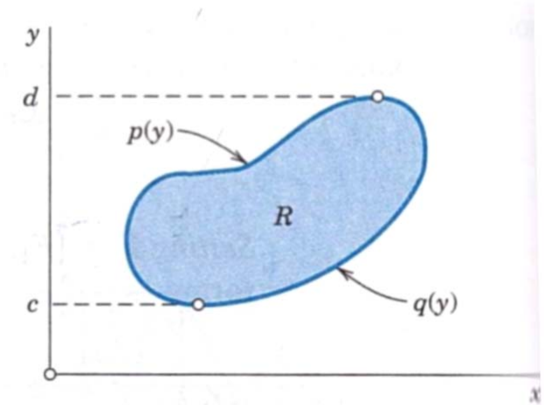
$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_C (Pdx + Qdy) \qquad (6.13.1)$$



Proof:

We'll divide the proof of Green's theorem in two stages.

1. At the first stage, let's prove the theorem for a special region R whose boundary is a single closed path. Then



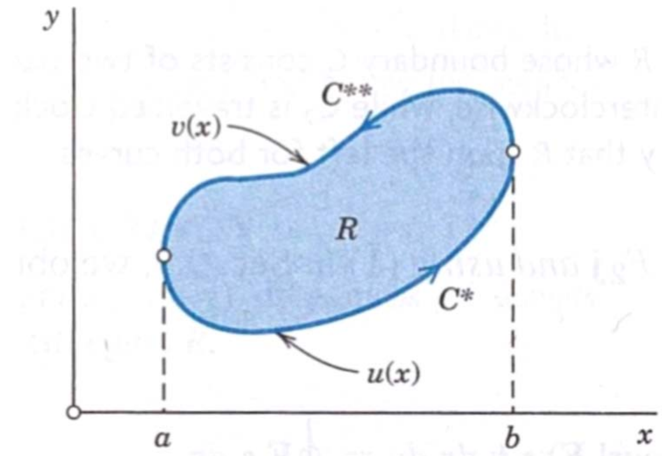
$$\iint_R \frac{\partial Q}{\partial x} dxdy = \int_c^d \left[\int_{p(y)}^{q(y)} \frac{\partial Q}{\partial x} dx\right] dy = \int_c^d [Q(q(y),y) - Q(p(y),y)]dy$$

$$= \int_c^d Q(q(y),y)dy + \int_d^c Q(p(y),y)dy = \oint_C Qdy$$

6.13. Green's theorem for a plane

$$-\iint_R \frac{\partial P}{\partial y} dxdy = \int_a^b \left[\int_{v(x)}^{u(x)} \frac{\partial P}{\partial y} dy \right] dx = -\int_a^b [P(x,u(x)) - P(x,v(x))] dx$$

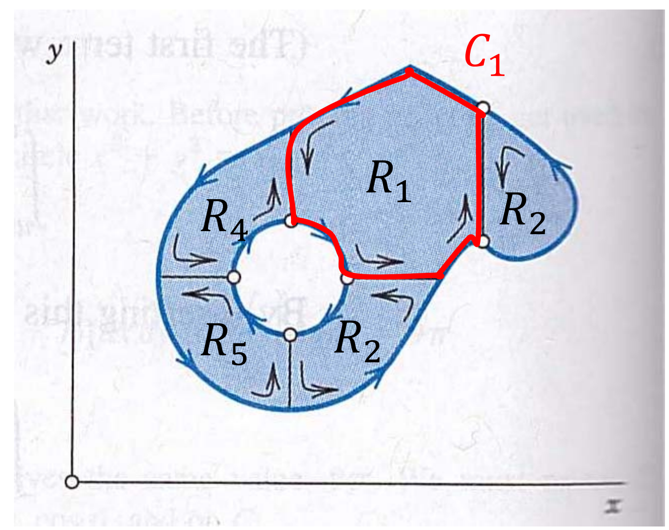
$$= \int_a^b P(x,v(x))dx + \int_b^a P(x,u(y))dy = \oint_C Pdx$$



The sum of two obtained equation gives us Eq. (6.13.1).

2. Now let prove Eq. (6.13.1) for arbitrary region whose boundary consists of a few closed paths. In this case the region R can be divided into a set of subregions R_1, R_2, etc., such that the boundary of every subregion is a simple closed path. Then we can apply already proved Green's formula to every subregion

$$\iint_{R_i} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint_{C_i} (Pdx + Qdy)$$



Sum of all these equation for all R_i gives us Green's formula for the whole region: On the left we have double integral for the whole region R (additivity), and on the right all parts of the paths that are counted twice will cancel each other due to their different directions.

6.13. Green's theorem for a plane

Consequence 1: $Pdx + Qdy$ is the full differential if and only if (recall first-order exact ODEs)

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

This is a particular case of the general theorem proved in the Section 6.14.

Consequence 2: Let $\mathbf{F}(x, y) = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$ be the vector field, which has a curl. Then

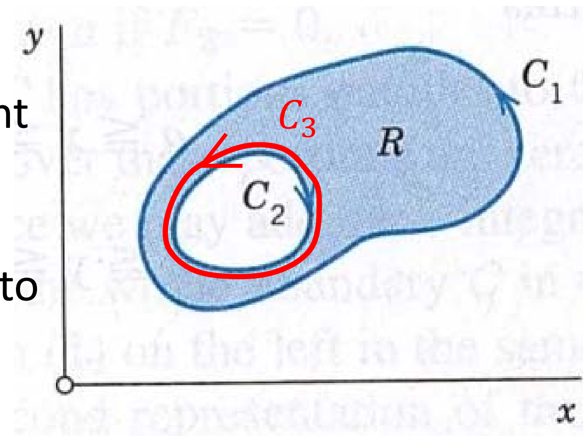
$$(\text{curl } \mathbf{F})_z = (\text{curl } \mathbf{F}) \cdot \mathbf{k} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

Then Green's theorem, Eq. (6.13.1), says that

$$\iint_R (\text{curl } \mathbf{F}) \cdot \mathbf{k} \quad dxdy = \oint_C (F_x dx + F_y dy) \qquad (6.13.2)$$

i.e. *circulation of the 2D vector field is equal to the double integral of the z-component of the curl over the region bounded by the integration path.*

Consequence 3: Consider two evenly oriented paths, which can be continuously transforms one to another in R. Then for a 2D gradient field, circulations these two paths are equal to each other if the region between them does not contain singular points of $\boldsymbol{F}$ (i.e. if $\mathbf{F}$ is continuous in R). Proof: Since curl $\mathbf{F} = 0$, Eq. (6.13.2) reduces to



$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0 \Longrightarrow \oint_{C_1} \mathbf{F} \cdot \boldsymbol{d}\mathbf{r} = \oint_{C_3} \mathbf{F} \cdot d\mathbf{r}$$

6.13. Green's theorem for a plane

Property $\oint_{C_1} \mathbf{F}\cdot d\mathbf{r} = \oint_{C_3} \mathbf{F}\cdot d\mathbf{r}$ does not hold even if curl **F** = 0 if

1. Field $\mathbf{F}$ has a singular point in R.
2. Paths cannot be continuously transform one to another.

These cases are very important for potential flows in fluid mechanics.

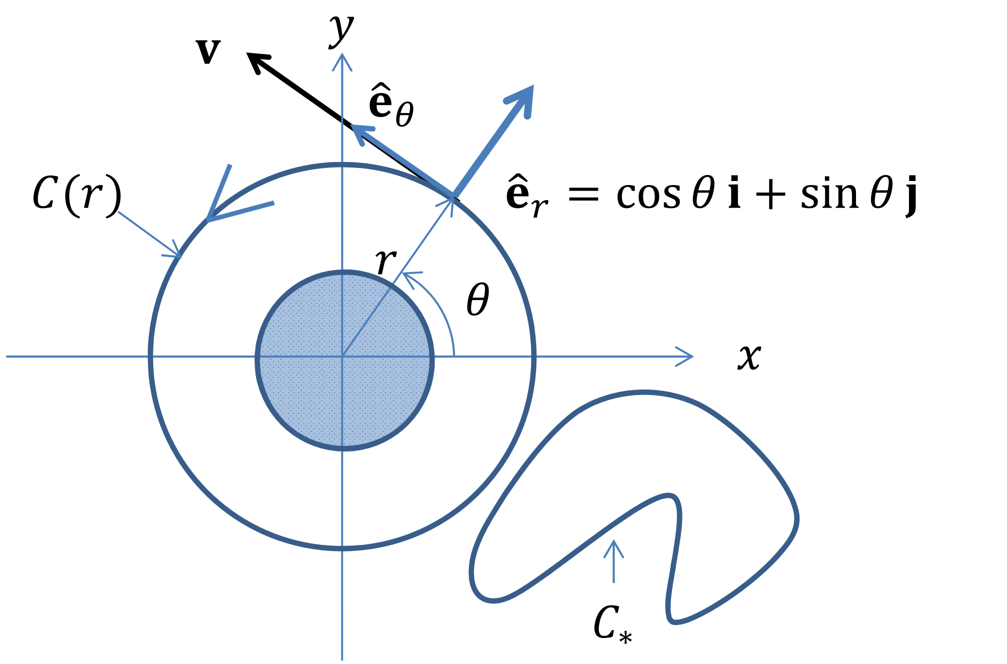
Example 1a: **2D inviscid vortex**: Flow field $\mathbf{v}$ with zero curl $\mathbf{v}$, but with non-zero circulation

Flow potential $\varphi = \frac{\Gamma}{2\pi}\arctan\frac{y}{x}$, Flow velocity $\mathbf{v} = \nabla\varphi = -\frac{\Gamma}{2\pi}\frac{y}{r^2}\mathbf{i} + \frac{\Gamma}{2\pi}\frac{x}{r^2}\mathbf{j}$

$$v_r = \mathbf{v}\cdot\hat{\mathbf{e}}_r = 0, \qquad v_\theta = \mathbf{v}\cdot\hat{\mathbf{e}}_\theta = \frac{\Gamma}{2\pi}\frac{1}{r}$$

$$\oint_{C(r)} \mathbf{v}\cdot d\mathbf{r} = \int_0^{2\pi} v_\theta\, r d\theta = \Gamma, \text{but} \quad \oint_{C_*} \mathbf{v}\cdot d\mathbf{r} = 0$$

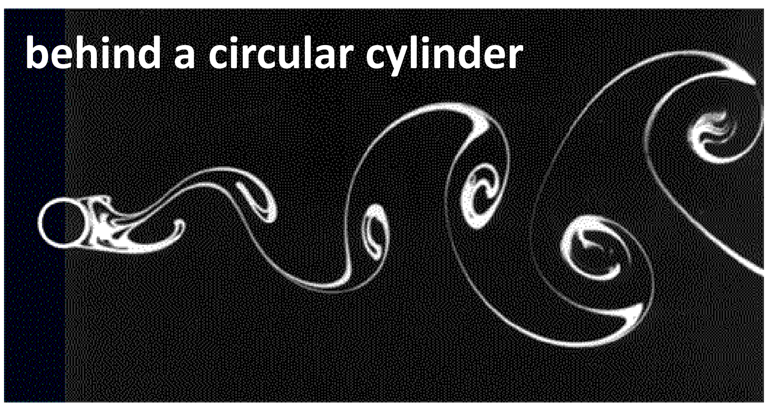
Example 1b: Irrotational flow potential flow over a cylinder with non-zero circulation: One can consider the flow field from the previous example in the domain R: $r \geq R_0$ outside a cylinder of radius R_0. Now R does not contain a singular point, but circulation over $C(r)$ is not equal to zero.



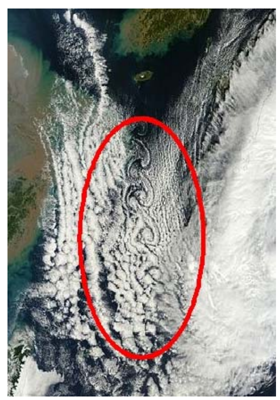
6.13. Green’s theorem for a plane

Vortexes develop in many natural and industrial flows:

Von Karman vortex street



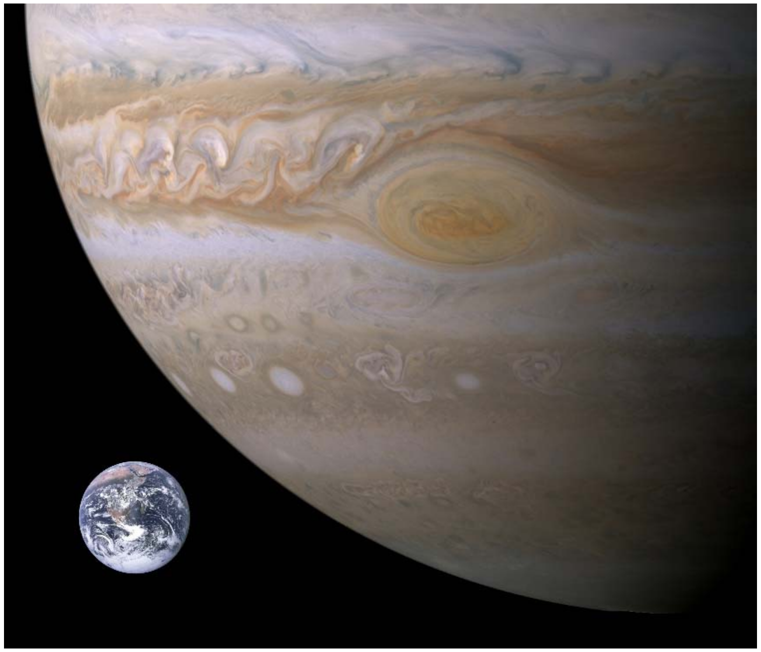
behind an island

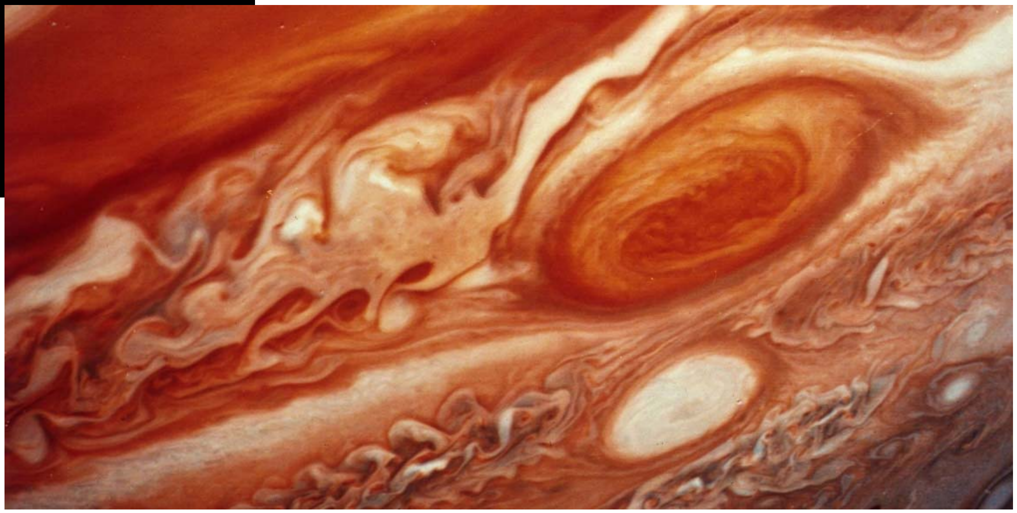


Flows can be considered as composed of discrete vortexes

6.13. Green's theorem for a plane

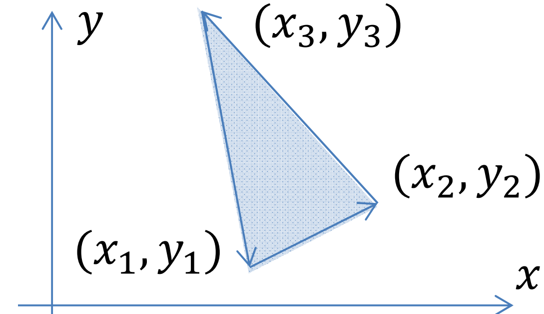
Red spot on Jupiter: a giant vortex





6.13. Green's theorem for a plane

Example 2: Calculation of the area of a region on xy-plane. If we introduce $P(x,y)$ and $Q(x,y)$ such that $\partial Q/\partial x - \partial P/\partial y = 1$, then the LHS in Green's theorem gives the area A of the region R. We can use, e.g. $Q = x$ and $P = 0$ or $Q = 0$ and $P = -y$. Then Green's theorem reduces to



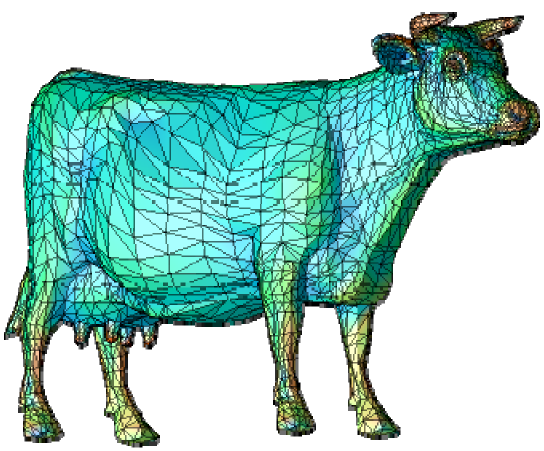
$$A = \oint_C x dy = -\oint_C y dx = \frac{1}{2}\oint_C (x dy - y dx)$$

Example 2a: Area of a triangle given by vertex coordinates

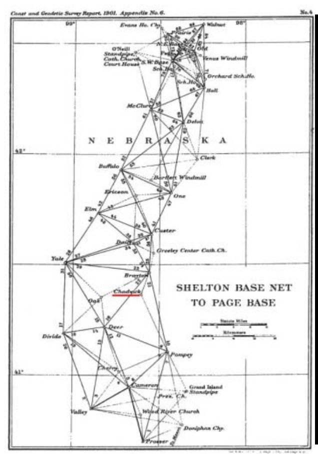
$$A = \oint_C x dy = \frac{1}{2}(x_2 + x_1)(y_2 - y_1) + \frac{1}{2}(x_3 + x_2)(y_3 - y_2) + \frac{1}{2}(x_1 + x_3)(y_1 - y_3)$$

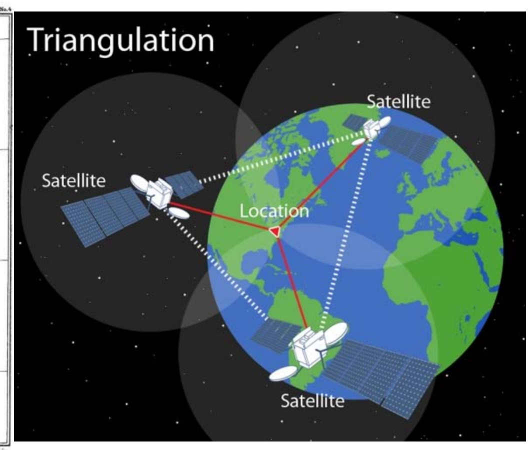
Calculation of geometrical properties of triangles is a routine operation in

Computer graphics

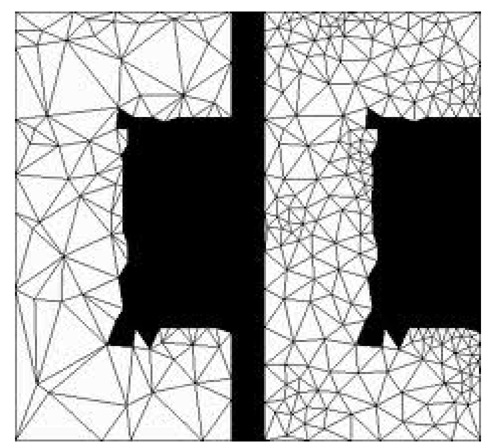


Mapmaking/GPS





Meshing domains by triangulation for numerical solution of PDEs



6.13. Green's theorem for a plane

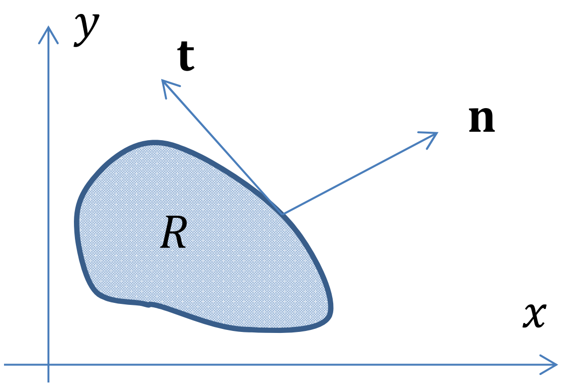
Consequence 4: Divergence theorem in 2D.

Assume that LHS in Eq. (6.13.1) is the divergence of some vector field $\mathbf{F}(x,y) = F_x\mathbf{i} + F_y\mathbf{j}$, i.e. $Q = F_x$ and $P = -F_y$, so $\partial Q/\partial x - \partial P/\partial y = \mathrm{div}\mathbf{F}$. Then in the RHS of Eq. (6.13.1) we have

$$\oint_C (Pdx + Qdy) = \oint_C \left(-F_y\frac{dx}{ds} + F_x\frac{dy}{ds}\right)ds = \oint_C (F_x n_x + F_y n_y)ds = \oint_C \mathbf{F}\cdot\mathbf{n}\,ds$$

where $\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j} = (dy/ds)\mathbf{i} - (dx/ds)\mathbf{j}$ is the **external unit normal** (normal directed to the exterior of R: if $\mathbf{t} = (dx/ds)\mathbf{i} + (dy/ds)\mathbf{j}$ is the tangent, then $\mathbf{n}$ is the normal since $\mathbf{t}\cdot\mathbf{n} = 0$).

Thus



(6.13.3) $$\iint_R \mathrm{div}\,\mathbf{F}\,dxdy = \oint_C (\mathbf{F}\cdot\mathbf{n})ds$$

Eq. (6.13.3) is the **divergence theorem in 2D**: Double integral of divergence over region R is equal to the line integral of the normal component of the vector field over the boundary.

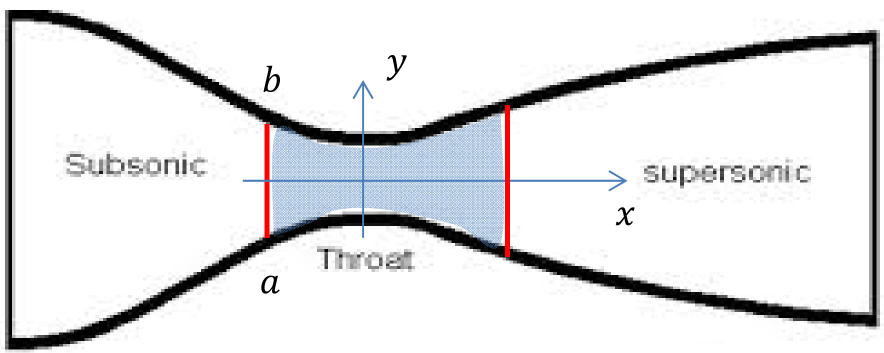
6.13. Green's theorem for a plane

Example 4: Mass flux through the plane nozzle.
In a steady-state flow, $\rho\mathbf{v}$ is a divergence-free field (ρ and $\mathbf{v}$ are fluid density and velocity):

$$\nabla \cdot (\rho\mathbf{v}) = 0$$

Then applying Eq. (6.13.3) to a region between two cross sections of a planar **Laval nozzle**, one can conclude that the mass flux $F(x)$ is constant for any cross section

$$F(x) = \int_{a(x)}^{b(x)} \rho(x,y)v_x(x,y)dy = const$$



6.13. Green's theorem for a plane (optional)

Example 4: Double integral of the 2D Laplacian.

In many application, the **Laplacian** of a scalar field $f(x, y)$ appears

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \mathrm{div}(\mathrm{grad}\, f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

For instance: The heat conduction equation in a medium with constant properties is

$$\rho c \frac{\partial T}{\partial t} = k \Delta T$$

Then we can apply our 2D divergence theorem:

$$\iint_R \Delta f dxdy = \iint_R \mathrm{div}(\nabla f) dxdy = \oint_C (\nabla f \cdot \mathbf{n}) ds = \oint_C \frac{\partial f}{\partial \mathbf{n}} ds \qquad (6.13.4)$$

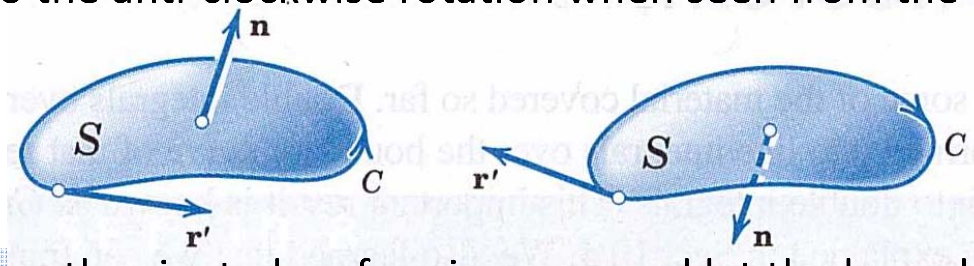
For 2D heat conduction equation:

$$\iint_R \rho c \frac{\partial T}{\partial t} dxdy = k \oint_C \frac{\partial T}{\partial \mathbf{n}} ds \quad \text{or} \quad \frac{d}{dt} \iint_R \rho c T(x, y, t) dxdy = -\oint_C \mathbf{q} \cdot \mathbf{n} ds \qquad (6.13.5)$$

where $\mathbf{q} = -k\nabla T$ is the heat flux. Eq. (6.13.5) is the mathematical representation of the energy **conservation law** for a 2D finite volume of a quiescent medium.

6.14. Stokes's theorem

Let's assume that we have an oriented surface S with normal $\mathbf{n}$, whose boundary is a simple closed curve with a given direction. We say that the orientation of the surface S and direction of C satisfy the **right hand-side screw rule** (or **corkscrew rule**) if motion along C in the positive direction corresponds to the anti-clockwise rotation when seen from the top of $\mathbf{n}$.



Stokes's theorem:

Let S be a piecewise smooth oriented surface in space and let the boundary C of S be piecewise smooth simple closed curve. Let $\mathbf{n}$ be a unit normal vector to S and orientation of S and direction of C satisfy the right hand-side screw rule.

Let $\mathbf{F}(x, y, z)$ be a continuous vector field that has continuous first partial derivatives in a domain in space containing S. Then

(6.14.1) $$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dA = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

or, in components

$$\iint_S \left[\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) dydz + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) dxdz + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dxdy \right] = \oint_C (F_x dx + F_y dy + F_z dz)$$

6.14. Stokes's theorem

Proof:

1. We can represent the vector field $\mathbf{F}$ in the form

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

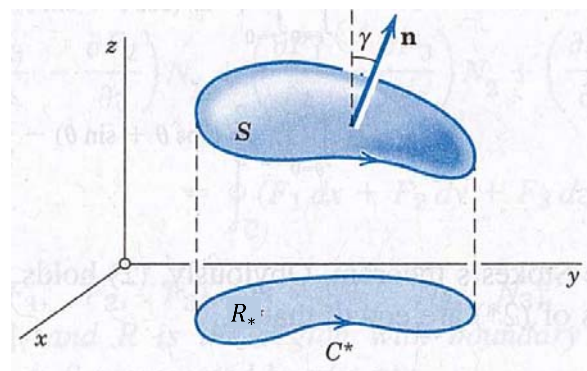
where $\mathbf{F}_1 = F_x\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$, etc. If we prove that Stokes's theorem holds for $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ individually, then it holds for $\mathbf{F}$ which is easy to prove using linearity of the curl and line and surface integrals. Obviously, it is enough to prove theorem only for $\mathbf{F}_1$, proofs for $\mathbf{F}_2$ and $\mathbf{F}_2$ will be analogues.

2a. First, let's prove the theorem for a special surface that can be represented in the form $z = f(x, y)$. The idea of the following proof is to reduce the problem to the Green's theorem. Setting $u = x$ and $v = y$, we have

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$$

$$\mathbf{t}_u = \mathbf{t}_x = \mathbf{i} + \frac{\partial f}{\partial x}\mathbf{k}, \qquad \mathbf{t}_v = \mathbf{t}_y = \mathbf{j} + \frac{\partial f}{\partial y}\mathbf{k}$$

$$\mathbf{N} = \mathbf{t}_u \times \mathbf{t}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \partial f/\partial x \\ 0 & 1 & \partial f/\partial y \end{vmatrix} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$$



$$\iint_S (\text{curl } \mathbf{F}_1) \cdot \mathbf{n}\, dA = \iint_{R_*} \left[\frac{\partial F_x}{\partial z} N_y - \frac{\partial F_x}{\partial y} N_z\right] dxdy = -\iint_{R_*} \left[\frac{\partial F_x}{\partial z}\frac{\partial f}{\partial y} + \frac{\partial F_x}{\partial y}\right] dxdy \quad (6.14.2)$$

Here R_* is the region on the xy-plane bounded by the curve C_* and we can apply Green's theorem for this region.

6.14. Stokes's theorem

Now let's calculate RHS in Eq. (6.14.1) for the vector field $\mathbf{F}_1$:

$$\oint_C \mathbf{F}_1 \cdot d\mathbf{r} = \oint_C F_x dx = \oint_{C*} F_x(x, y, f(x,y))dx = -\iint_{R*} \left[\frac{\partial F_x}{\partial y} + \frac{\partial F_x}{\partial z}\frac{\partial f}{\partial y}\right] dxdy \quad (6.14.3)$$

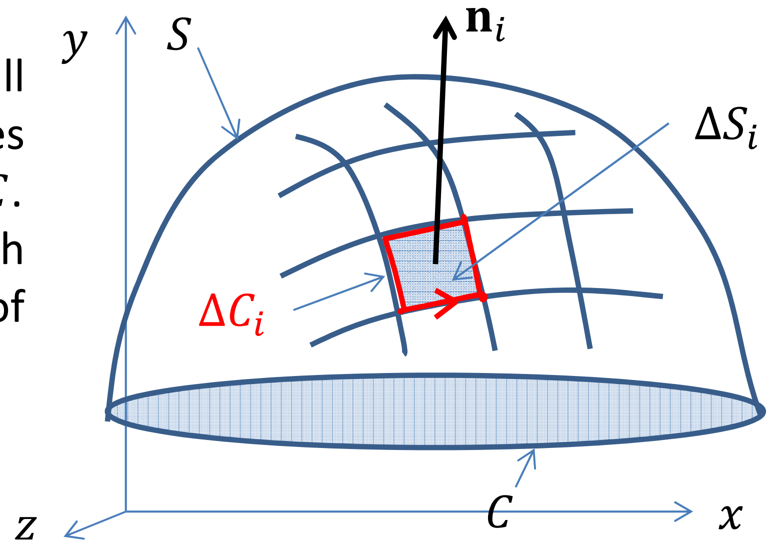
Now we see that the RHSs in Eqs. (6.14.2) and (6.14.3) are the same and it proves Stokes's theorem for $\mathbf{F}_1$ in the case of a special surface represented in the form $z = f(x, y)$.

2b. Now let's consider a surface that can be divided into a finitely many parts, where every part can be represented in the form $z = f(x, y)$. Then we can apply the previous proof to every individual part and then use the additivity of the surface and line integrals.

3. In order to proof the theorem for an arbitrary surface, we need to remember definitions of the surface and line integrals as the sum limits:

1. We can divide the surface S into large number of small and almost planar cells - surfaces ΔS_i bounded by curves ΔC_i keeping for every i orientation of the original S and C.
2. For every subsurface, considering it as plane with normal $\mathbf{n}_i$, we can apply Green's theorem in the form of Eq. (6.13.2):

$$\iint_{\Delta S_i} (\text{curl } \mathbf{F}) \cdot \mathbf{n}_i dA = \oint_C \mathbf{F} \cdot d\mathbf{r} \quad (6.14.4)$$



3. The sum of Eqs. (6.14.4) for all cells in the limit $\max(\Delta l_i) \to 0$ gives Eq. (6.14.1), since the line integrals along all "internal" edges of cells cancel each other.

6.14. Stokes's theorem

Consequence 1: Criterion for the path independence of the line integral.

Reminder: Line integral is said to be **path independent** in a domain D if for every pair of endpoints A and B in the domain D, the line integral has the same value for all paths in D that begin in A and end at B. We have proved in Section 6.9 that the following statements are equivalent:

1. Line integral for $\mathbf{F}$ is path independent.
2. $\mathbf{F}$ is a gradient field.
3. Any circulation of $\mathbf{F}$ is zero.
4. Differential form (6.14.2) for $\mathbf{F}$ is exact

$$\mathbf{F} \cdot d\mathbf{r} = F_x dx + F_y dy + F_y dz \qquad (6.14.4)$$

Now let's find the criterion for exactness of (6.14.4) based on Stokes's theorem:

Theorem: Criterion of exactness and path independence.

Assume that components of $\mathbf{F}$ have continuous first derivatives in D. Then

1. If the differential form (6.14.2) is exact, then curl $\mathbf{F} = 0$, i.e.

$$\frac{\partial F_z}{\partial y} = \frac{\partial F_y}{\partial z}, \qquad \frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}, \qquad \frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y} \qquad (6.14.5)$$

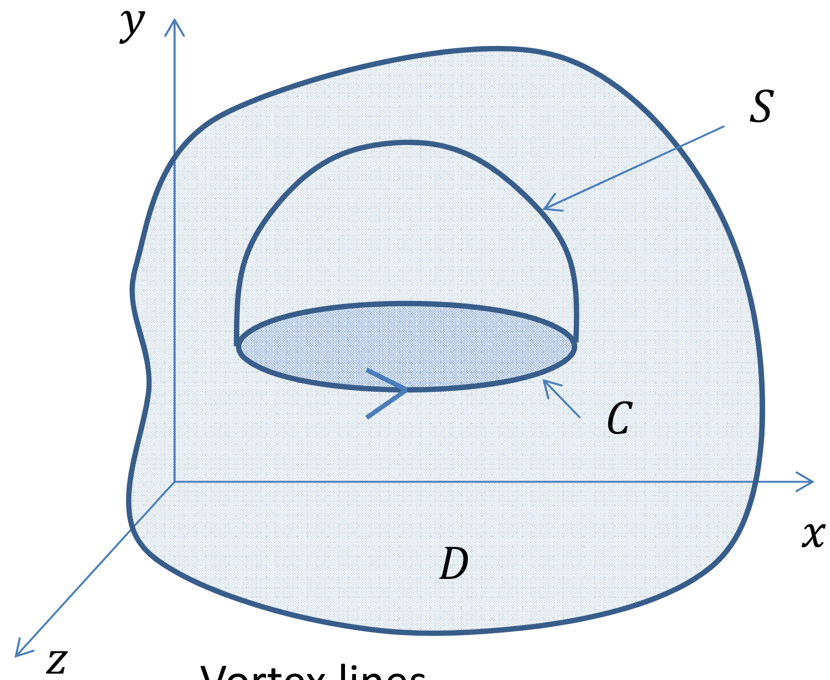
2. If curl $\mathbf{F} = 0$ (i.e. (6.14.5) holds) in D and domain D is simply connected, then (6.11.4) is exact in D.

6.14. Stokes's theorem

Proof:

1. if (6.14.4) is exact, then $\mathbf{F} = \nabla f$, and thus, $\mathrm{curl}\,\mathbf{F} = 0$ since it holds for any gradient field.

2. Let C be a closed path in D. Since D is simply connected, we can find S in D bounded by C. If $\mathrm{curl}\,\mathbf{F} = 0$, then, according to Stokes's theorem, circulation of $\mathbf{F}$ is zero for any closed path in D. Thus, line integral for $\mathbf{F}$ is path independent and the form given by Eq. (6.11.2) is exact (see theorems in Section 6.6).

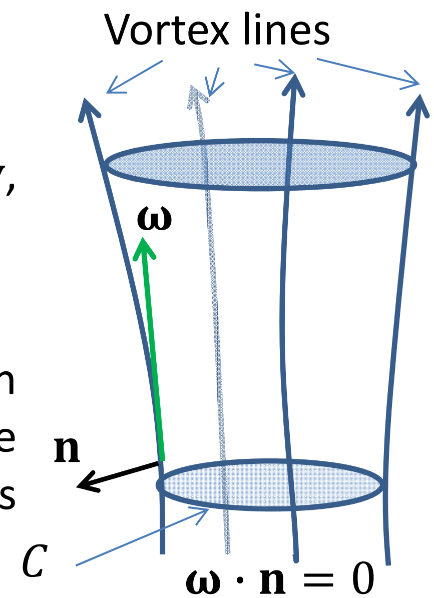


Consequence 2: Helmtholtz's theorems for vortex tubes

Let's consider the vector field of **vorticity** $\boldsymbol{\omega} = \mathrm{curl}\,\mathbf{v}$, where $\mathbf{v}$ is the fluid velocity field.

Field lines of $\boldsymbol{\omega}$ are called the **vortex lines**.

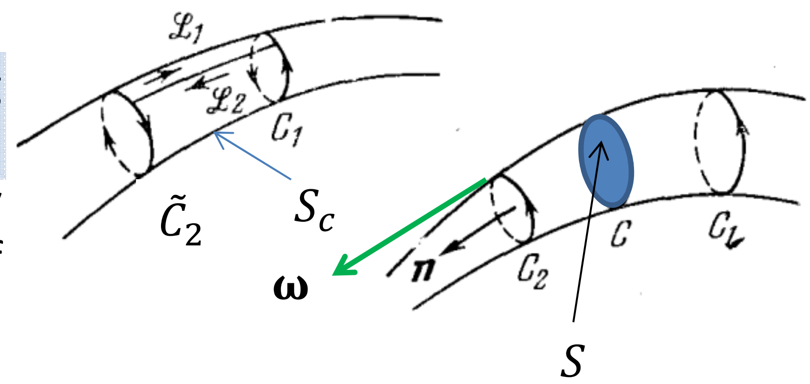
Let's consider some closed curve C not coinciding with any vortex line. The **vortex tube** for the curve C is the region bounded by all vortex lines going through points on the curve C.



6.14. Stokes's theorem

1st Helmholtz's theorem:

Circulation of fluid velocity over any contour bounding the vortex tube is constant.

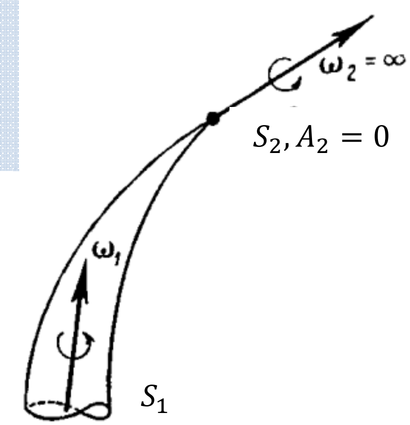


Proof: Let's choose two contours C_1 and C_2 and apply Stokes's theorem to the closed contour composed of C_1, $\tilde{C}_2$, $\mathcal{L}_1$, and $\mathcal{L}_2$ (S_c corresponds to the lateral surface of the tube between C_1 and C_2):

$$\oint_C \mathbf{v}\cdot d\mathbf{r} = \iint_{S_C} (\mathrm{curl}\,\mathbf{v})\cdot\mathbf{n}\,dA = 0,\ \oint_{C_1} \mathbf{v}\cdot d\mathbf{r} + \oint_{\tilde{C}_2} \mathbf{v}\cdot d\mathbf{r} = 0,\ \oint_{C_1} \mathbf{v}\cdot d\mathbf{r} = \oint_{C_2} \mathbf{v}\cdot d\mathbf{r} = \Gamma = const$$

2nd Helmholtz's theorem:

If $\mathbf{v}$ and $\boldsymbol{\omega}$ are continuous fields, then a vortex tube cannot end in a fluid, it must extend to the boundaries of the fluid or form a closed path.



Proof: According to the 1st Helmholtz's theorem

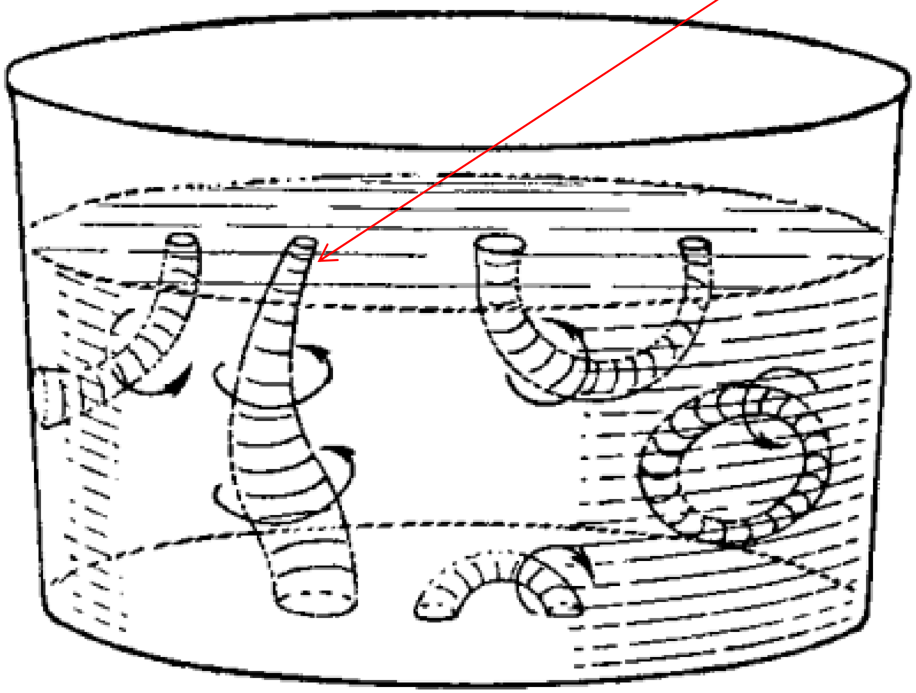
$$\oint_C \mathbf{v}\cdot d\mathbf{r} = \iint_S (\mathrm{curl}\,\mathbf{v})\cdot\mathbf{n}\,dA = \Gamma = const$$

If the tube ends inside the fluid, the length of the circumference of C (and surface area of S) at the tube end becomes zero and velocity and vorticity will be infinite. Thus, $\mathbf{v}$ and $\boldsymbol{\omega}$ can not be continuous at the end.

6.14. Stokes's theorem

Consequence of the 2nd Helmholtz's theorem: Tornado

Vortex tubes



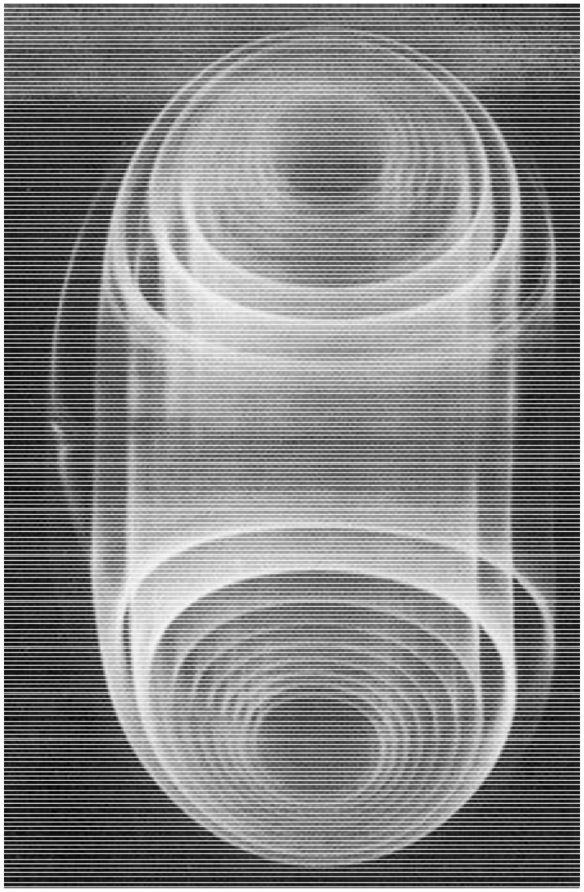
$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{v}) \cdot \mathbf{n} \, dA = \Gamma = const$$

- Tornado starts at the surface.
- At the surface: Minimum circumference, maximum fluid velocity.
- Long lifetime, propagation to large distances

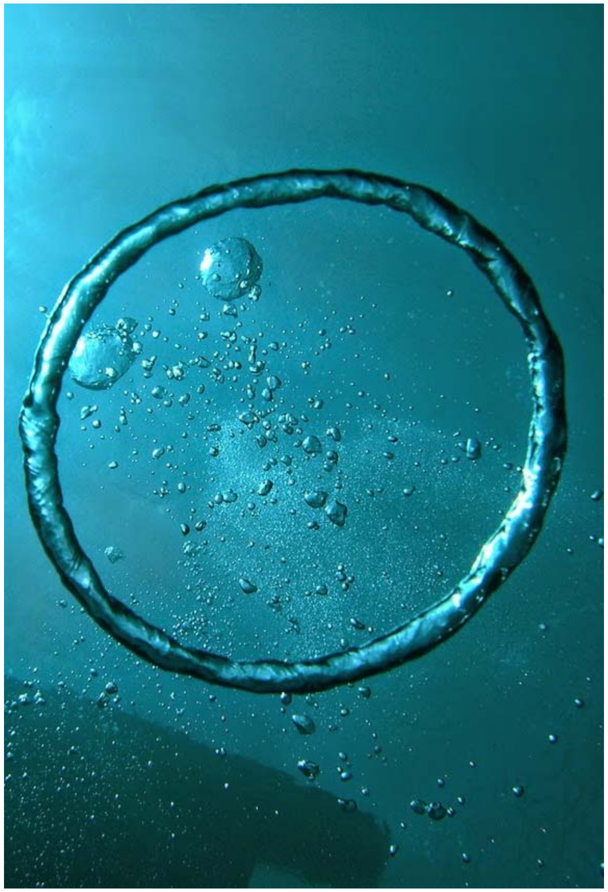
Consequence of the 2nd Helmholtz's theorem: Vortex ring and mushroom vortex

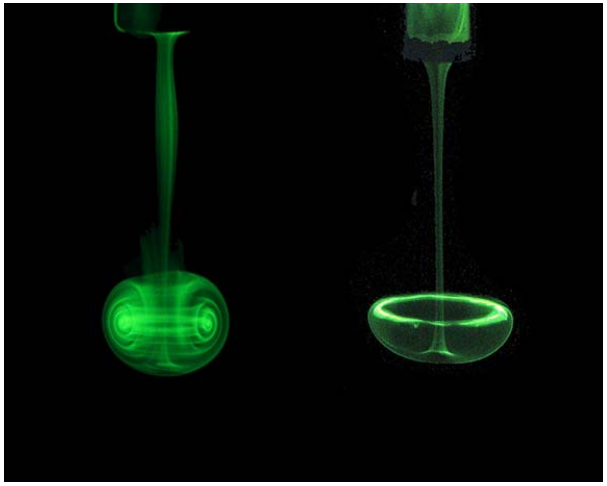
Vortex ring is the toroidal ("donut-type") closed vortex tube

Smoke in air



Water in air





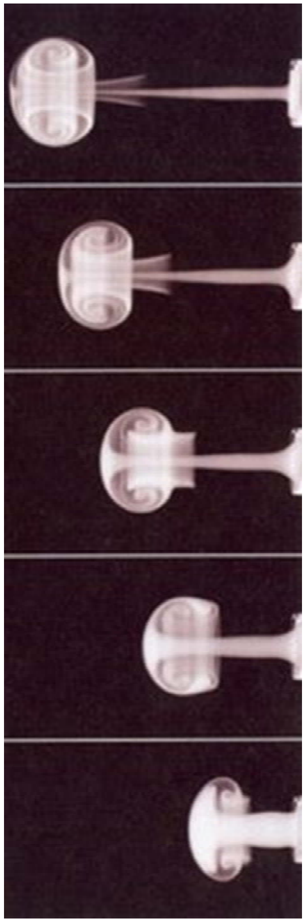
6.14. Stokes's theorem

Consequence of the 2nd Helmholtz's theorem: Vortex gun

As long as the effects of viscosity and diffusion are negligible, the fluid in a moving vortex is carried along with it. In particular, the fluid in the core (and matter trapped by it) tends to remain in the core as the vortex moves about. *Thus vortices (unlike surface and pressure waves) can transport mass, energy and momentum over considerable distances compared to their size, with surprisingly little dispersion.*

See https://en.wikipedia.org/wiki/Vortex.

Momentum transfer to large distances by vortex rings is used in **vortex guns**.



6.15. Gauss divergence theorem

Gauss divergence theorem gives a connection between the triple integral of the divergence of a vector field over a closed bounded domain and the surface integral of this vector field over the surface of this domain.

Gauss divergence theorem:

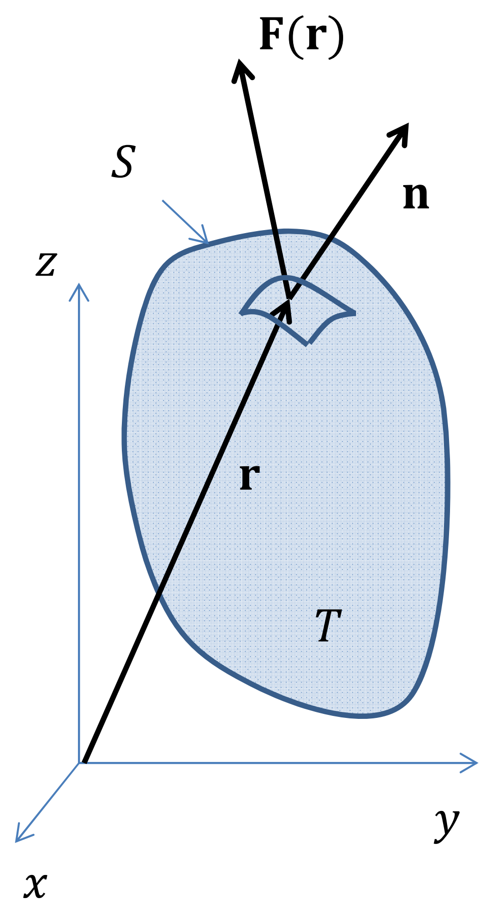
Let T be a closed bounded region in 3D space whose boundary is a peecewise smooth orientable surface S. Let $\mathbf{F}(\mathbf{r})$ be a vector field that has continuous components and continuous partial derivatives of components in some domain containing T. Then

(6.15.1) $$\iiint_T \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dA$$

where $\mathbf{n}$ is external unit normal surface vector with respect to T, or, in components

$$\iiint_T \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dxdydz = \iint_S (F_x dydz + F_y dxdz + F_z dxdy)$$

$$= \iint_S (F_x \cos\alpha + F_y \cos\beta + F_z \cos\gamma)\, dA$$



6.15. Gauss divergence theorem

Proof:

The Gauss divergence theorem can be easily proved for special domain T which is simple with respect to x, y, and z (see section 4.7). Then every triple integral of an individual term, e.g. $\partial F_x / \partial x$, can be reduced by successive calculation to the double integral that finally reduces to the surface integral of F_x. See the proof in Kreyszig's textbook, pages 454-456.

Another approach is to divide T into a set of small cells and apply Eq. (6.8.6) to every cell. Then the sum over all individual cells in the limit of infinitely small cell gives the Gauss theorem.

Application of Gauss divergence theorem

The major application of the Gauss divergence theorem in mechanics and heat transfer is the derivation of the differential equations based on **conservation laws initially formulated in the integral form**, i.e. applied to a finite volume of medium.

- Heat conduction equation
- Navier-Stokes and Euler equations in fluid mechanics
- Maxwell equations of electromagnetic field

6.15. Gauss divergence theorem

Example 1: Derivation of differential heat conduction equation

Conservation law for internal energy in a quiescent medium:

$$\frac{d}{dt}\iiint_T \rho c T \, dV = -\iint_S \mathbf{q} \cdot \mathbf{n} \, dA \qquad (6.15.2)$$

Rate of change of internal energy in T — Flux of internal energy through the boundary of T

Now we can apply the Gauss divergence theorem:

$$\iint_S \mathbf{q} \cdot \mathbf{n} \, dA = \iiint_T \operatorname{div} \mathbf{q} \, dV \quad \Rightarrow \quad \iiint_T \rho c \frac{\partial T}{\partial t} \, dV = -\iiint_T \operatorname{div} \mathbf{q} \, dV$$

$$\iiint_T \left(\rho c \frac{\partial T}{\partial t} + \operatorname{div} \mathbf{q} \right) dV = 0 \qquad (6.15.3)$$

This equation should be valid for any T. It is possible **only** if

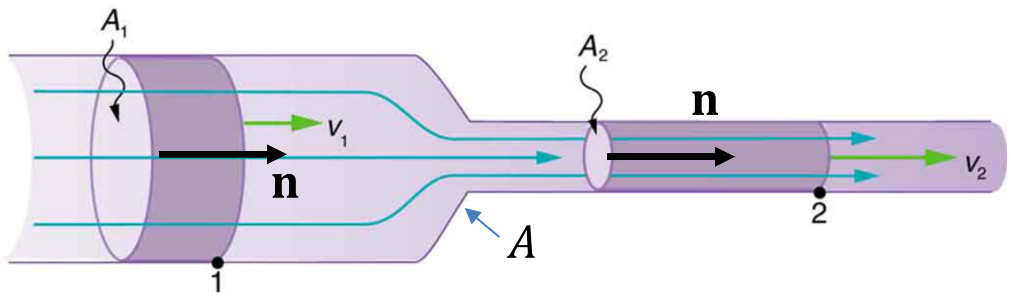
$$\rho c \frac{\partial T}{\partial t} + \operatorname{div} \mathbf{q} = 0$$

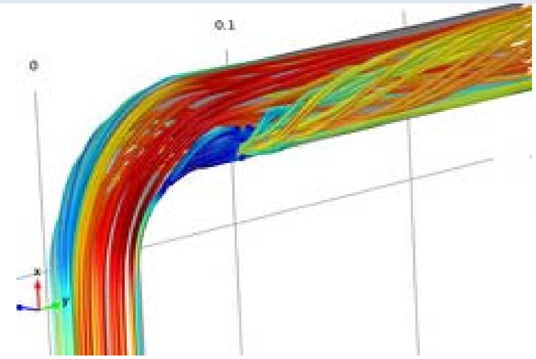
Eq. (6.15.3) is the heat conduction equation (energy equation in a quiescent medium).

This is the differential form of the energy conservation law (1st law of thermodynamics).

6.15. Gauss divergence theorem

Example 2: Flow of incompressible fluid through a pipe.





For incompressible fluid (density $\rho = const$), the velocity field if divergence free:

$$\text{div}\,\mathbf{v} = 0$$

Let's integrate this equation over a volume T between two cross sections A_1 and A_2:

$$\iiint_T \text{div}\,\mathbf{v}\,dV = 0$$

This integral is equal to zero, because the pipe walls are **impenetrable** for the fluid and thus $\mathbf{v}\cdot\mathbf{n} = 0$

Let's apply the Gauss theorem:

$$\iiint_T \text{div}\,\mathbf{v}\,dV = \iint_S \mathbf{v}\cdot\mathbf{n}\,dA = \iint_{A_1} \mathbf{v}\cdot\mathbf{n}\,dA + \iint_A \mathbf{v}\cdot\mathbf{n}\,dA - \iint_{A_2} \mathbf{v}\cdot\mathbf{n}\,dA = 0$$

Then the mass flux (**mass flow rate**) Q through any cross-section of the tube will be the same no matter how complex the flow velocity field is:

$$Q = \rho \iint_{A_1} \mathbf{v}\cdot\mathbf{n}\,dA = \iint_{A_2} \mathbf{v}\cdot\mathbf{n}\,dA = const$$